THE

PHYSICAL SOCIETY

OF

LONDON.

PROCEEDINGS.

VOLUME XXVIII.—PART IV.

JUNE 15, 1916.

Price to Non-Fellows, **4s.** *net, post free* **4/3**.

Annual Subscription, **20/-** *post free, payable in advance.*

Published Bi-Monthly from December to August.

LONDON:
"THE ELECTRICIAN" PRINTING & PUBLISHING CO., LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET

1916.

THE PHYSICAL SOCIETY OF LONDON.

1916-17.

OFFICERS AND COUNCIL.

President.

PROF. C. VERNON BOYS, F.R.S.

Vice-Presidents.

Who have filled the Office of President.

PROF. G. C. FOSTER, D.Sc., LL.D., F.R.S.
PROF. R. B. CLIFTON, M.A., F.R.S.
PROF. A. W. REINOLD, C.B., M.A., F.R.S.
SIR W. de W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.
PRIN. SIR OLIVER J. LODGE, D.Sc., LL.D., F.R.S.
PROF. SILVANUS P. THOMPSON, D.Sc., F.R.S.
R. T. GLAZEBROOK, C.B., D.Sc., F.R.S.
PROF. J. PERRY, D.Sc., F.R.S.
C. CHREE, Sc.D., LL.D., F.R.S.
PROF. H. L. CALLENDAR, M.A., LL.D., F.R.S.
PROF. A. SCHUSTER, Ph.D., Sc.D., F.R.S.
SIR J. J. THOMSON, O.M., D.Sc., F.R.S.

Vice-Presidents.

W. R. COOPER, M.A., B.Sc.
F. E. SMITH.
S. W. J. SMITH, M.A., D.Sc., F.R.S.
W. E. SUMPNER, D.Sc.

Secretaries.

W. ECCLES, D.Sc.,
26, Ridgmount Gardens, W.C.

R. S. WILLOWS, M.A., D.Sc.,
Cass Technical Institute, Jewry Street, Aldgate, E.C.

Foreign Secretary.

R. T. GLAZEBROOK, C.B., D.Sc., F.R.S.

Treasurer.

W. DUDDELL, F.R.S.
56, Victoria Street, S.W.

Librarian.

S. W. J. SMITH, M.A., D.Sc., F.R.S.
Imperial College of Science and Technology.

Other Members of Council.

H. S. ALLEN, M.A., D.Sc.
PROF. W. H. BRAGG, M.A., F.R.S.
S. D. CHALMERS, M.A.
PROF. G. W. O. HOWE, D.Sc.
PROF. J. W. NICHOLSON, M.A., D.Sc.
C. C. PATERSON.
C. E. S. PHILLIPS, F.R.S.E.
PROF. O. W. RICHARDSON, M.A., D.Sc., F.R.S.
G. F. C. SEARLE, Sc.D., F.R.S.
F. J. W. WHIPPLE, M.A.

XVIII.—*The Laws of Variation of Resistance with Voltage at a Rectifying Contact of Two Solid Conductors, with Applications to the Electric Wave Detector.* *By* D. Owen, *B.A., B.Sc., Birkbeck College, London.*

Received March 20, 1916.

Various theories have been put forward to account for the rectifying action at a contact such, for example, as that between zincite and chalcopyrite, to which an alternating voltage is applied. It is not necessary to refer to more than two, which alone have up to the present claimed serious consideration. Both before and since Dunwoody's application of the mineral detector to wireless telegraphy, it was recognised that certain minerals possess extraordinarily large thermoelectric powers, exceeding in some instances by ten times or more that between bismuth and antimony. A natural explanation in terms of thermoelectric action thus lay to hand, which appeared all the more probable from the fact that these minerals also possess a large negative temperature coefficient of resistance.

In 1907, however. appeared a Paper by Prof. G. W. Pierce, containing an account of experiments which were regarded as precluding such an explanation, the main ground for this conclusion being the undoubted fact that the direction of flow of the rectified current across a junction is generally (in the case of mineral contacts) contrary to that of the current set up on heating the same junction.

A theory—since known as the electronic valve theory—was thrown out in that Paper, according to which we have in certain minerals a new phenomenon, namely, the easy passage of free electrons in one direction across the contact, but not in the contrary direction. This is the second view which has held the field.

The thermoelectric explanation has been developed by Dr. W. Eccles, who in 1913 expressed it in mathematical form, and showed the concordance of this theory with many of the observed phenomena.

On the other hand, despite the attraction which the free electronic view has possessed for many writers, little, if anything, has been done to establish definite laws answering to that view; and this tempting hypothesis has not advanced beyond the stage of mere speculation.

Most workers on this subject have directed their attention

to the determination of the current characteristic, both current and voltage being directly measured. This method has several defects : it involves the application of the voltage for a comparatively long interval of time ; again it is least accurate where the results are of the greatest interest, namely, at very small voltages. Current characteristics are usually taken with applied voltages up to or beyond 1 volt, and some workers have extended the range up to 20 volts. The minimum voltage which is necessary to produce a signal in the receiving circuit of a wireless station is probably only of the order of a hundredth of a volt.

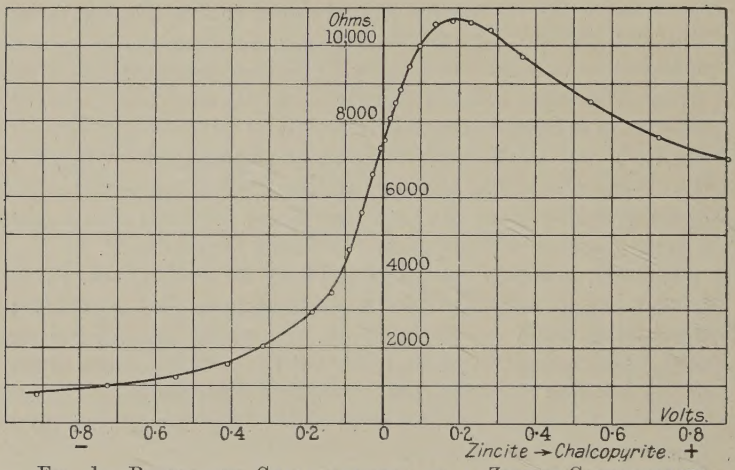

Fig. 1.—Resistance Characteristic of a Zincite-Chalcopyrite Contact.

The present Paper deals with measurements of the resistance of the contact. The primary aim was to examine to what extent the resistance-characteristic expresses the nature of the physical actions occurring, and to obtain data leading to a discrimination between the hypotheses referred to.

Preliminary measurements showed that with care to avoid mechanical disturbance, and provided time was allowed for the contact to " settle down "—a process sometimes occupying minutes, sometimes hours—the resistance was sufficiently definite and reliable to permit of the application of the Wheatstone bridge, with all the power and flexibility of this method of attack.

The characteristic obtained with a zincite-chalcopyrite contact is shown in Fig. 1.

This type was followed by all the mineral combinations tried. The main points of interest are as follows : When a small voltage is applied the resistance rises, falling at the same rate when the voltage is reversed ; a maximum occurs on the side of initial increase ; at the higher voltages the characteristic is a falling one, the gradient diminishes on either side, positive or negative. Let

v=voltage applied across the contact.
R_0=resistance when v is zero.
R_m=maximum value of resistance, occurring at voltage v_m.

Investigation of a great number of contacts has led to the following quantitative results :—

1. At $v=0$ the gradient of the resistance characteristic is proportional to the limiting resistance R_0, that is, $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$ is a constant for a given pair of materials. This quantity, which is analogous to the temperature coefficient of resistance, may be termed the voltage coefficient of the contact ;

2. v_m is independent of R_0 ;

3. The ratio R_m/R_0 is independent of R_0. When a series of characteristics are obtained from contacts of widely varying resistance or area of contact, and the ordinates of each reduced in the ratio $1 : R_0$, the resulting curves are found to coincide, yielding what may be called the *specific characteristic* of the given pair of substances (Figs. 2 and 10).

The above results are found to conform to an expression connecting R and v derived on the basis that the phenomena are thermoelectric.

A series of experiments was carried out on a number of contacts both elements of which were metallic, and for which the thermoelectric and other relevant constants are known. These confirm the same view.

The facts as to the sign of the rectified current will be shown on close examination to be, not in opposition, but in simple accordance with the thermoelectric explanation.

The experimental results above outlined form the basis of a calculation of the conditions of best sensibility in a wireless receiving circuit, with or without a polarising voltage. Similar considerations determine the sensitiveness to be expected in a combination of rectifying contact and direct-current galvanometer, as applied to the detection of small alternating voltages in bridge or other null methods of measurement.

I. *Experimental Details.*

The contacts tested were found to range in resistance from fractions of an ohm in the case of metallic combinations to millions of ohms in the case of zincite or carborundum contacts. The Wheatstone bridge was employed throughout. The applied voltage was varied from a millivolt or less up to 1 or 2 volts, the value being read on a multiple-range millivoltmeter in shunt with the bridge. If P and Q denote the ratio arm resistances, S that of the variable arm, V the voltage across the bridge, then the voltage v applied to the contact, and its apparent resistance R are calculated from the formulæ,

$$v=\frac{P}{P+Q}\,.\,V, \text{ and } R=\frac{P}{Q}\,.\,S.$$

The current through the contact is, of course, given by the expression v/R.

The two elements of the contact were—in the case of minerals—soldered by means of Wood's fusible metal into copper cups hard-soldered on to copper plates. In order to damp mechanical vibration the contact was mounted on a thick pad of cotton wool. A simple lever arrangement permitted of continuous adjustment of the mechanical pressure at the contact. Non-rigid electrical connection was obtained by the use of mercury cups wherever necessary.

The effect of time of application of the voltage was found to be considerable at the higher voltages. By limiting the duration of the testing current the measured resistance becomes quite determinate. Small periods of contact were obtained by aid of a metallic ball or ring suspended by a fine wire which also served as battery lead. On pulling out by a fine thread, then letting go, the ball was made to strike a heavy metal plate, thus making the circuit for the period of collision, which was determined in a separate experiment. Many of the measurements recorded were taken with a contact lasting 0·003 second. In order to secure the requisite sensibility under these conditions a Broca galvanometer of 1,000 ohms resistance was used.

In all the measurements the galvanometer key was kept closed, the resistance in the variable arm of the bridge being adjusted to give balance on closing the battery circuit.

The sign of the thermo E.M.F. was found for each contact in situ, the small rise of temperature required being obtained without any disturbance of the contact by applying a flame

momentarily to a thick copper wire attached to the plate which supported the contact.

II. *The Experiments on Minerals.*

The mineral contacts tested were the following :—

Zincite-platinum.	Galena-platinum.
Zincite-chalcopyrite.	Chalcopyrite-platinum.
Carborundum-platinum.	Bornite-platinum.

Chalcopyrite-bornite.

It was considered desirable from the point of view of simplicity and reproducibility to use a metal as an element of the contact. Platinum was chosen on account of its unoxidisability and high melting point.

Zincite.—Early observations showed that remarkably steady resistances were obtainable with the zincite-chalcopyrite combination. Thus, tests were conducted on the same contact at intervals extending over a week without any variation of the limiting resistance greater than 5 per cent. After a series of measurements over a range up to 1 volt, lasting about an hour and a half, the initial resistance was reproduced often to within a few parts in a thousand. To secure such constancy it is necessary to avoid excessive voltages—say, more than 1·5 volt—or at all events to apply them for only a small fraction of a second.

The following example of readings refers to the test from which the resistance characteristic of Fig. 1 was drawn.

TABLE I.—*Zincite-Chalcopyrite.*

P=1,000, Q=100.

v	R	Direction of current.	v	R	Direction of current.
0·0045	7,490	Zinc—>Chalc	0·0045	7,300	Chalc—>Zincite
0·0236	8,170	,,	0·091	4,620	,,
0·0463	8,910	,,	0·136	3,520	,,
0·0682	9,530	,,	0·182	3,000	,,
0·091	10,030	,,	0·227	2,600	,,
0·136	10,650	,,	0·318	2,100	,,
0·186	10,750	,,	0·409	1,640	,,
0·232	10,680	,,	0·545	1,300	,,
0·0045	7,500	,,	0·727	1·040	,,
0·282	10,500	,,	0·919	830	,,
0·364	9,800	,,	0·0045	7,350	,,
0·545	8,650	,,	...	...	...
0·727	7,700	,,	...	...	...
0·91	7,100	,,	...	...	...
0·0045	7,600	,,	...	...	...

Ten different contact points were taken in succession between the same pair of mounted minerals, and sets of readings taken from which the resistance characteristics were plotted. The limiting resistances were found to vary from a few thousands to a few millions of ohms. The contacts were chosen so as to give values of resistance approximately equispaced over the whole range, otherwise no selection was exercised. Variation of the mechanical pressure serves conveniently to secure the above adjustment. The highest value used was 540,000 ohms, since for higher values the diminishing sensibility of the bridge becomes unfavourable to accuracy at the lowest voltages. The following table summarises the results of the series :—

TABLE II.—*Series of Contacts Between Zincite and Chalcopyrite.*

No.	R_0	R_m	R_m/R_0	v_m	$(dR/dv)_0$	$\left(\frac{1}{R}\frac{dR}{dv}\right)_0$
1	6,480	10,100	1·50	0·17	27,700	6·70
2	7,360	10,800	1·47	0·18	32,800	4·40
3	20,340	30,600	1·47	0·17	113,800	5·07
[4	24,000	28,100	1·17	0·30	39,500	1·64]
5	57,200	83,000	1·45	0·15	296,000	5·17
6	77,200	104,200	1·35	0·12	405,000	5·24
7	80,800	132,000	1·63	0·16	591,000	7·32
8	123,000	202,500	1·62	0·14	925,000	7·40
9	251,500	445,000	1·77	0·18	$2·18\times10^6$	8·70
10	540,000	855,000	1·58	0·11	$4·1\times10^6$	7·6
			Mean =1·54	Mean =0·15		Mean =6·40

$(dR/dv)_0$ is in every case found to be positive for current from zincite to chalcopyrite across the contact. With the exception of No. 4 (which was excluded in calculating mean values), it will be seen that the variations in the three quantities v_m, R_m/R_0 and $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$ are insignificant compared with the range of resistance covered by the series. When the *a priori* probability of lack of homogeneity in the material is considered, the variation of individual results from the mean for the nine contacts is surprisingly small. It seems safe to conclude that the essential action at the contact is expressed in the statement that $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$, R_m/R_0, *and* v_m *are dependent only on the materials constituting the contact, and not on the area or resistance of the contact itself.*

An examination of the curves indicates that at every point (unless v is large) the ratio R/R_0 is independent of the value of R_0. The graphs were reduced in scale so as to make R_0 unity in each case. The specific characteristics thus obtained for the zincite-chalcopyrite and other mineral contacts are shown in Fig. 2.

Galena.—A similar series of measurements was undertaken on contacts between platinum and galena. The specific resistance of the latter is much smaller than that of zincite, so that very small mechanical pressure at the contact must be used in

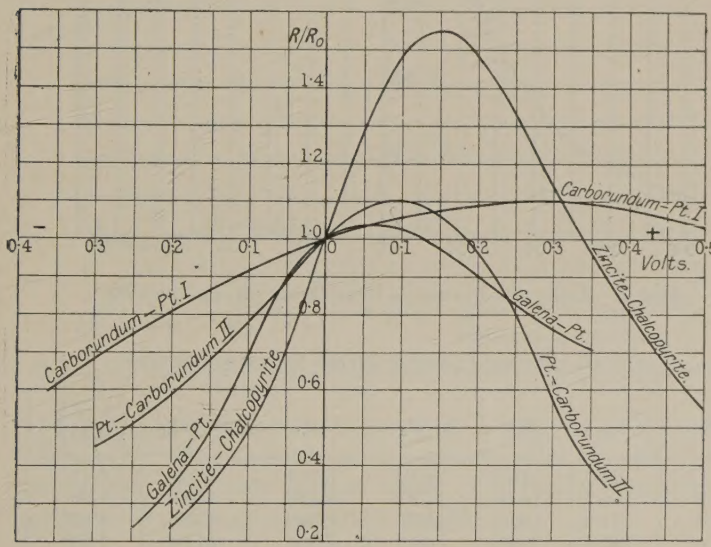

FIG. 2.—SPECIFIC RESISTANCE-CHARACTERISTICS OF SOME MINERAL CONTACTS.

order to secure even moderately high values of the resistance. A platinum wire of No. 24 S.W.G. was found suitable. The pressure was supplied by the spring action of this wire, or rather of a copper wire which terminated in a short length of the platinum wire. It is not easy to work with these light contacts, and much time and care were necessary to secure a satisfactory set of readings. Ten contacts were examined, the lowest being 34·1 ohms and the highest 1,400 ohms. Attempts at measurement on still higher values, a finer platinum wire being substituted, were abandoned as hopeless.

One specimen curve is given in Fig. 3. The summarised results are contained in Table III.

$(dR/dv)_0$ is in every case positive when the current crosses the contact from galena to platinum. The conclusion to be

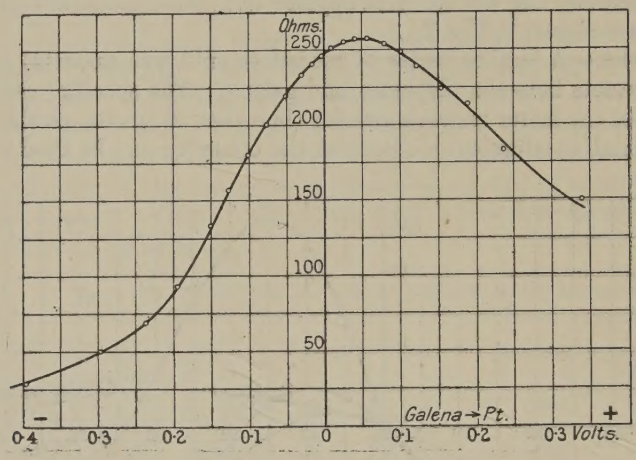

FIG. 3.—A GALENA-PLATINUM RESISTANCE CHARACTERISTIC.

TABLE III.—*Series of Contacts between Galena and Platinum.*

No.	R_0.	Rm.	Rm/R_0.	vm.	$(dR/dv)_0$.	$\left(\frac{1}{R}\frac{dR}{dv}\right)_0$.
1	34·1	34·5	1·012	0·061	15·5	0·45
2	116·7	140·0	1·05	0·077	140·0	1·20
3	132·8	139·2	1·048	0·066	172·5	1·30
4	247·4	258·0	1·047	0·055	411·0	1·66
5	299·0	308·0	1·027	0·060	365·0	1·22
6	440·0	454·0	1·032	0·067	322·0	0·73
7	570·0	593·0	1·04	0·068	503·0	0·88
8	750·0	781·0	1·04	0·072	1,200·0	1·60
9	1,305·0	1,390·0	1·065	0·067	2,320·0	1·78
10	1,368·0	1,436·0	1·056	0·070	1,650·0	1·20
			Mean =1·042	Mean =0·066		Mean =1·20

drawn is again that the quantities $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$, v_m, and R_m/R_0 are independent of the limiting resistance of the contact. The voltage coefficient is about one-fifth the value of the same quantity for zincite-chalcopyrite.

Carborundum.—A contact of this material exhibits the same

general behaviour, as is shown in the specimen curve of Fig. 4. Seven contacts were investigated, five on one specimen, which consisted of a conglomerate of small crystals (Series C), and

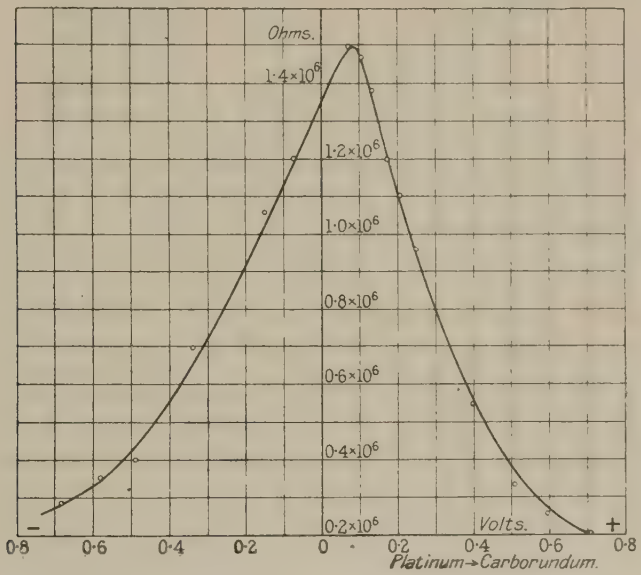

FIG. 4.—A HIGH-RESISTANCE PLATINUM-CARBORUNDUM CHARACTERISTIC.

two on another specimen, bounded by a large facet of crystal (Series C′). The resistances of contacts over the latter were never less than two million ohms, even under mechanical pressure as great as a kilogram weight.

TABLE IV.—*Carborundum-Platinum Contacts.*

No.	R_0.	R_m.	R_m/R_0.	v_m.	$(dR/dv)_0$.	$\left(\frac{1}{R}\frac{dR}{dv}\right)_0$.	Direction of current for $\left(\frac{dR}{dv}\right)_0$ positive.
Series C :—							
1	460	510·5	1·11	0·28	370	0·86	Carb. to plat.
2	525	590·0	1·12	0·26	550	1·05	,, ,,
3	612	680·0	1·11	0·30	401	0·65	,, ,,
4	923	953·0	1·04	0·30	206	0·22	,, ,,
5	198,000	225,000	1·14	0·16	562,500	2·81	Plat. to carb.
Series C′ :—							
1	$1·34\times10^6$	$1·5\times10^6$	1·12	0·075	$2·0\times10^6$	1·49	Plat. to carb.
2	$2·05\times10^6$	$2·6\times10^6$	1·04	0·04	$3·6\times10^6$	1·44	,, ,,

Omitting No. 5 of Series C, the following mean values may be assigned :—

Series.	$\left(\frac{1}{R}\frac{dR}{dv}\right)_0$.	v_m.	R_m/R_0.	Direction of current for $\left(\frac{dR}{dv}\right)_0$ positive.
C	0·67	0·285	1·10	Carb. to plat.
C′	1·46	0·057	1·08	Plat. to carb.

It will be seen that four of the contacts were of low resistance (under a thousand ohms), and the remaining three of high resistance. In the former the resistance *rises* for currents increasing from zero in the direction carborundum to platinum across the contact, whilst in the latter the resistance *falls* for currents in that same direction. Tests of the thermo-E.M.F.s obtained on heating the contacts showed that the thermo-electric power of the high resistance contacts was opposite in sign to that of the low resistance contacts. Apart from this striking evidence of heterogeneity of structure, the contacts of carborundum exhibit the same characteristic behaviour as galena and zincite, and thus fall within the same category of phenomena.

Other Minerals.—Experiments on the combinations of platinum-chalcopyrite and chalcopyrite-bornite show that the sensitiveness of these contacts is negligibly small compared with that of the preceding combinations. In the case of chalcopyrite-bornite three contacts were tested, the values of R_0 being respectively 1,952, 1,880 and 1,500 ohms. The corresponding voltage-coefficients at $v=0$ were 0·073, 0·05 and 0·00. Three contacts of platinum-chalcopyrite had resistances 160, 473 and 793 ohms respectively at $v=0$; the corresponding voltage-coefficients being 0·00, 0·00 and 0·026. A few contacts of platinum-zincite were also tested; they yielded values of $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$, R_m/R_0, and v_m, practically agreeing with those of chalcopyrite-zincite.

We may conclude that chalcopyrite, bornite and platinum are almost interchangeable elements of a combination whose other member belongs to minerals of the class zincite or carborundum. Bornite has the advantage of possessing a lower specific resistance than chalcopyrite.

The Thermo-electric Sign.—This was determined for all the above minerals in the manner indicated at the end of Section I. One element was platinum in each case.

TABLE V.—*Thermo-electric Sign of Minerals against Platinum.*

Material.	Direction of current across heated junction.
Zincite	Zincite to *Pt.*
Chalcopyrite	Chalcopyrite to *Pt.*
Bornite	*Pt* to bornite.
Galena	Galena to *Pt.*
Carborundum (low resistance)	Carborundum to *Pt.*
Carborundum (high resistance)	*Pt* to carborundum.

The characteristics shown in the Paper are drawn in accordance with the above results, the applied voltage being treated as positive when producing current across a junction in the same direction as that caused by a slight rise of temperature at that junction.

III. *The Thermo-electric Theory of the Phenomena.*

The contacts under investigation are such that on the one side the current passes across an area which may be of the order of a square centimetre, whilst on the other side it crosses an area many thousands of times smaller. Regarding the combination as a thermo-electric element, there will be a disengagement of heat on two accounts: Joulean heat, proportional to the square of the current and irreversible in sign; and Peltier heat, proportional to the current, and, therefore, reversed in sign when the current is reversed. The changes of temperature will be negligibly small at the junction of large area in comparison with that at the minute contact. We may, therefore, confine our attention to the latter only.

When the applied voltage is large—say, half a volt or higher—the heat at the contact is sufficiently great to cause appreciable softening, with consequent increase of contact area and drop of resistance. This effect supervenes earlier for current in one direction across the contact than when reversed.

The effects are simple provided the applied voltage is restricted to a sufficiently small value. In the account that follows this limitation is assumed.

Consider in the first place the case of a contact having zero temperature coefficient of resistance. If the Joulean heat could be neglected a rise of temperature would occur on applying a voltage in one direction, and an equal fall with voltage reversed. Equal and opposite thermo-E.M.F.s will be generated, the direction being such as in each case to

oppose the applied voltage. The apparent resistance of the contact will be greater than the true resistance, the excess being independent of the direction of the current. Its value will be constant, since for small voltages the back thermo-E.M.F. will be proportional to the applied voltage. Now let us correct this result by taking the Joulean heat into account. For one direction of current this will be added to the Peltier heat, thereby increasing the rise of temperature and the back E.M.F., and consequently the apparent resistance. For current in the opposite direction the Joulean heat will partially neutralise the Peltier cooling; and the fall of temperature, the back thermo-E.M.F. and the apparent resistance will all be reduced. The resultant effect will be to give a straight line law connecting apparent resistance and applied voltage, the line being now inclined to the axis of voltage.

Another elementary thermodynamic principle of importance here is that when current across the junction causes Peltier *absorption* of heat, an external supply of heat must cause an E.M.F., and, therefore, a current, in the *same* direction across that junction. It follows that at a junction of materials possessing zero temperature-coefficients of resistance the apparent resistance *falls* when the current is of the same sign as that caused by heating the junction. Such a case is found in the combination manganin-constantan, observations on which are described below.

Let us next proceed to the case in which the specific resistances of the elements forming the contact vary appreciably with temperature. If the temperature coefficient is positive the effect will be to augment both the fall of resistance for one direction of current, and the rise for current in the reverse direction. The slope of the resistance characteristic in the immediate neighbourhood of zero voltage will be of the same sign as before, but increased in value.

When the temperature coefficient is negative, on the other hand, the slope of the characteristic at zero voltage is reduced, and changes sign when the coefficient is sufficiently large numerically, in which case we have a *rise* of apparent resistance when the current is in the same direction as that caused by applying heat to the junction. It will be shown that the large negative coefficients occurring in minerals like zincite and galena are amply sufficient to produce this rise. Thus is explained the apparent anomaly of a reduction in resistance despite the fact of a back thermo-E.M.F., and, vice versa, of a

rise in resistance although the thermo-E.M.F. is in the direction to assist the applied E.M.F.

Attention may be further directed to the importance of the part played by a large negative temperature coefficient of resistance. Neglect of this consideration has given rise to fantastic views of the rises of temperature which must be invoked to account for actual resistance variations. Take, for example, the case of a zincite-chalcopyrite contact, at which was observed a fall of resistance from 20,000 ohms at zero voltage to 500 ohms under 1 volt. If temperature variation of resistance is neglected we are faced with the necessity of supposing a thermo-E.M.F. e in the direction of the applied voltage, such that $(1+e)$ volts across 20,000 ohms would produce the same current as 1 volt across 500 ohms—that is, e must be 39 volts. Assuming the thermo-electric power to be 1,000 microvolts per degree Centigrade, and independent of temperature, the required rise of temperature is 39,000°C. But if we suppose, on the other hand, zero thermo-electric power and a temperature variation of resistance at the rate of $-1{\cdot}5$ per cent. per 1°C., then assuming a compound interest law we have $20{,}000\ (1-0{\cdot}015)^t=500$, whence the rise of temperature is 243°C.; and we are back in the region of probability as well as possibility.

In working out an exact treatment of the actions which are assumed to take place at a contact the author is indebted to the Papers of Dr. W. Eccles * on this subject. Nevertheless the conclusions of the present Paper will be found to differ from those of the Paper quoted in several material respects.

Thus, whereas much importance is attached in the latter to the part played by the Thomson effect, in the present treatment the experimental results find their explanation in terms of the Peltier effect and the temperature coefficient resistance.

Again, the softening effect already referred to produces a considerable variation of resistance at the higher voltages, and this fact limits the range of voltage within which calculations can be made. This limitation renders unnecessary the supposition of a "dead resistance"; the elimination of the coefficient representing this allows of still further simplification and increased definitions in the final expressions obtained.

* Eccles, "Proc." Phys. Soc., Lond., XXV., p. 273, 1913.

Let—

T=Absolute temperature at the contact.
T_0=Absolute temperature of the surroundings.
$t=T-T_0$=rise of temperature.
a=Thermo-electric power.
e=Thermo-E.M.F.
P=Peltier effect=aT.
i=Current.
v=Voltage.
$R=v/i$=apparent resistance.
R'=True resistance.
b=Temperature coefficient of true resistance.
H=Total rate of heat developed at contact.
K=Cooling coefficient.

Both v and i will be considered to be positive when in the direction of the current due to a slight heating of the contact.

Assuming the thermo-E.M.F. to be a linear function of temperature change,

$$e=at. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking the rate of removal of heat from the contact to be proportional to temperature rise, then in the steady state,

$$H=Kt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

R and R' are related as follows :—

$$\frac{v}{R}=\frac{v+e}{R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Assuming a linear variation of R' with temperature—

$$R'=R_0'(1+bt). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The total heat developed at the contact by a current i is the sum of the Joule and Peltier effects. Observing the convention above defined for the positive direction of current, we have

$$H=i^2R'-Pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

On eliminating t, T, e, i, H, and P by aid of these relations we obtain the following expression for the apparent resistance of the contact in terms of the applied voltage :—

$$KR^2-R(KR_0'+a^2T_0)=v^2bR_0'-av(R+bT_0R_0'). \quad . \quad . \quad (6)$$

This is the equation of the resistance-characteristic. Three

main types of characteristic are derivable, which are represented diagrammatically in Fig. 5.

Type I. is the limiting case where $b=0$, when equation (7) simplifies to a straight line, denoting a resistance which falls for voltage applied in the positive direction. It is practically realised in the contact constantan-manganin.

Type II. includes all cases where b is positive, and thus covers all combinations of pure metals. With voltage in the positive direction the resistance at first falls, reaches a minimum, and afterwards rises.

Type III. represents cases for which b is negative and large. It includes all the mineral contacts possessing marked recti-

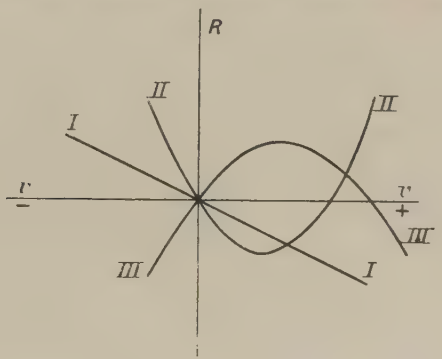

FIG. 5.—THREE THEORETICAL TYPES OF RESISTANCE CHARACTERISTIC.
Type I, $b=0$. Type II, b positive. Type III, b negative and large.

fying properties. The resistance *rises* at first with a positive voltage, reaches a maximum, and then falls continuously.

We now proceed to an examination of the values of $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$, R_m/R_0, and v_m which, it has been concluded from experimental observation, are constant quantities for any contact between the same pair of materials. It will now be shown that this constancy results from the theory, *provided* that the product KR_0 is a constant for the given pair of materials.

Putting $v=0$, equation (6), reduces to

$$R_0=R_0'+\frac{a^2T_0}{K}=R_0'\left(1+\frac{a^2T_0}{KR_0}\right). \quad . \quad . \quad . \quad . \quad (7)$$

The apparent resistance is greater than the true resistance by an amount proportional to the square of the thermo-electric

power. Calculation shows that in the case of metals the term $\frac{a^2T_0}{KR_0'}$ is negligible compared with unity, and it is probable that in the case of minerals also its value is less than unity. Approximate results may, therefore, be obtained by neglecting the term in a^2 in equation (6).

Differentiating (6), we have

$$\frac{dR}{dv}=\frac{2bR_0'v-a(R+bT_0R_0')}{2KR-KR_0'-a^2T_0+av}. \quad . \quad . \quad . \quad (8)$$

There are two important special cases. First, let $v=0$. Then

$$\left(\frac{dR}{dv}\right)_0=\frac{-a(R_0+bT_0R_0')}{KR_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or

$$\left(\frac{1}{R}\frac{dR}{dv}\right)_0=\frac{-a(1+bT_0R_0'/R_0)}{KR_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The expression in (10) represents the value of the voltage-coefficient of resistance of the contact. Its value is independent of R_0 provided that KR_0 is a constant.

In the next place let us consider the point at which dR/dv is equal to zero, that is, R a maximum or minimum. The applied voltage at which this occurs is given by

$$v_m=\frac{a(R_m+bT_0R_0')}{2bR_0'}=\frac{a(R_m/R_0'+bT_0)}{2b}. \quad . \quad . \quad . \quad . \quad (11)$$

Substituting this expression for v in the general equation, we obtain a function of the form $f(R_m/R_0, KR_0)=0$, which means that provided KR_0 is independent of R_0, so also is R_m/R_0.

Returning to equation (11), and putting R_m/R_0' equal to a constant, we see that v_m necessarily becomes independent of R_0.

It follows, therefore, that on the thermo-electric theory of the effect the three quantities $\left(\frac{1}{R}\frac{dR}{dv}\right)_0$, v_m, and R_m/R_0 are independent of R_0, provided KR_0 is independent of R_0.

The Value of KR_0.—In seeking for a theoretical basis yielding the requirement that KR_0 should be a constant, or approximately a constant, for any given pair of materials, two views of the nature of the contact have been taken. From either of these the required condition can be deduced.

(*a*) We may assume that the contact resistance is given by the end-correction formula applied to, say, a uniform column of mercury in a capillary tube terminating within a considerable bulk of mercury. Applying Lord Rayleigh's formula to a

contact supposed circular, of radius r, and denoting the specific resistance of the badly conducting component of the contact by ρ, we have

$$R_0' = \frac{0{\cdot}8r}{\pi r^2}\rho = \frac{0{\cdot}252\rho}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the next place let us assume the heat to be developed symmetrically within a spherical volume whose linear dimensions are of the order of r of (12), and that the amount set free per unit volume at a point distant r from the centre of the sphere is proportional to r^n, where n is some constant. Then total heat developed per second within the sphere of radius r_0

$$= H = \alpha \int_0^{r_0} 4\pi r^2 \cdot r^n dr = \frac{4\pi\alpha {r_0}^{n+3}}{n+3} \quad . \quad . \quad . \quad . \quad (13)$$

where α is a constant.

In the steady state the heat crossing any isothermal surface is equal to the heat liberated within it. Denoting the thermal conductivity by k,

$$-4\pi r^2 k \frac{dt}{dr} = \frac{4\pi\alpha r^{n+3}}{n+3},$$

whence

$$\frac{dt}{dr} = \frac{-\alpha r^{n+1}}{k(n+3)}.$$

Hence for the excess of temperature t' of the centre over the boundary of the sphere we have

$$t' = \frac{\alpha}{k(n+3)} \int_0^{r_0} r^{n+1} dr = \frac{\alpha {r_0}^{n+2}}{k(n+2)(n+3)}. \quad . \quad . \quad (14)$$

Similarly the excess temperature t'' of the boundary of the sphere of radius r_0 over that of the surroundings is found to be given by

$$t'' = \frac{\alpha {r_0}^{n+2}}{k(n+3)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Adding t' and t'' the excess temperature t_0 of the centre of the sphere over that of the surroundings has the value

$$t_0 = \frac{\alpha {r_0}^{n+2}}{k(n+2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Since $H = Kt_0$, we find from (13) and (16)

$$K = H/t_0 = 4\pi k r_0 (n+2)/(n+3) \quad . \quad . \quad . \quad . \quad (17)$$

Now, according to (18) $K \propto r_0$; whilst by (12) $R_0' \propto \frac{1}{r_0}$. Combining (12) and (17),

$$KR_0' = 0{\cdot}8k\rho(n+2)/(n+3) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The value of this expression depends upon the ratio of the thermal and electrical conductivities, and is independent of the dimensions of the contact.

This same conclusion may also be reached if we assume

(*b*) That the resistance R_0 is located mainly—let us suppose entirely—in a thin film of molecular dimensions whose area is that over which the two elements of the contact intimately meet. There is an *a priori* improbability that the resistance of a mere contact such as is here under consideration can be calculated exactly as though the materials were welded in perfect molecular union across the minute area of the contact—which is really what is assumed in the view (*a*). According to the second view, to the resistance calculated on the lines of (*a*) should be added a resistance inversely proportional to the area of contact.

Assuming this latter contribution to be the main part of the total resistance, it follows that the Joulean heat, as well as the Peltier heat, is located in a thin film of molecular dimensions at the interface of the two materials. The stream lines of heat and of electricity will be similar in form. We shall have K proportional to the area of contact, and R_0 inversely proportional to the area of contact. KR_0 will thus be independent of the contact area, and proportional to the ratio of the thermal and electrical conductivities.

The Slope of the Resistance-Characteristic when $v=0$.—Equations (9) and (10) indicate how the gradient at zero voltage depends on the value of the temperature coefficient of resistance.

When b is positive, $(dR/dv)_0$ must be negative. The resistance assumes a minimum at the positive voltage v_m. This case, as well as the limiting case when $b=0$, is the subject of the experiments described in section IV.

When b is negative, the gradient becomes positive if $(1+bT_0R_0'/R_0)$ is negative, or since R_0'/R_0 differs little from unity, when $-b > \frac{1}{T_0}$. Now, in the case of minerals b is negative and often much larger numerically than for a pure metal. Thus, Pierce, in a careful determination, obtained the value

$-0{\cdot}0153$ for molybdenite at 0°C. There is thus an ample margin for concluding from thermo-electric theory that $(dR/dv)_0$ should be positive at mineral contacts, so that theory and observation are in agreement as to the direction of the rectified current under an alternating voltage.

IV. *Experiments with Metallic Contacts.*

Experiments on metallic combinations are of interest, since they supply the case where b is positive or zero. Moreover, the values of a and b are definitely known. The following were selected as test cases :—

Bismuth-Antimony.—a large, $b+$.
Iron-Nickel.—a fairly large, $b+$.
Constantan-Manganin.—a fairly large, $b=0$.
Copper-Manganin.—a very small, $b\fallingdotseq 0$.
Bismuth-Bismuth.—a zero, $b+$.

Attention was specially directed to the junctions bismuth-antimony and constantan-manganin. A series of contacts was examined in each of these cases, the results being to show (1) that the voltage-coefficient at zero voltage is independent of the resistance ; (2) that the sign of the above coefficient is negative ; (3) that a minimum occurs on the side of positive voltage.

We may conclude that the phenomena occurring at metallic contacts are thermo-electric in nature, and are essentially identical with those of mineral contacts.

The voltage coefficient is largest in the case antimony-bismuth, though even here it has only one-fifth the value found at a galena-platinum contact, and about one twenty-fifth that of a zincite contact. The specific characteristics of the above two contacts are shown in Fig. 10, which is drawn to the same scale as Fig. 2.

Softening of the Contact due to Rise of Temperature.—All the resistance-characteristics of this class exhibit a droop on both the positive and the negative side when the voltage rises to a certain value dependent on the duration of the current. It occurs most readily in contacts containing bismuth as one element, the melting point of which is 269°C. The effect is evidently due to softening and plastic yielding at the contact, which results in rapid increase of area. There is little doubt that this effect is present to an important extent in the case

of mineral contacts also, though not so obvious since it then occurs on a falling characteristic.

Bismuth-Antimony.—An example of a characteristic is shown in Fig. 6. In order to obviate or diminish the softening

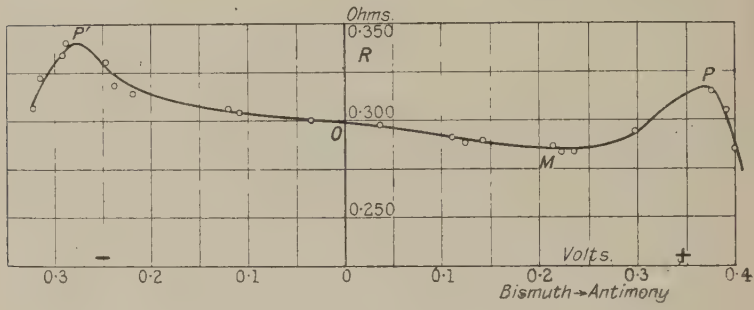

Fig. 6.—An Antimony-Bismuth Resistance Characteristic.

effect the voltage was applied for only 0·003 second. The droop then occurs when v is between 0·3 and 0·4 volt. Four contacts were examined. The results of Table VI. show the constancy of the voltage-coefficient.

Table VI.—*Bismuth-Antimony Contacts.*

No.	1.	2.	3.	4.	—
R	0·092	0·50	10·5	145·4	
(dR/d_0)	−0·019	−0·13	−2·75	−42·5	
$\left(\frac{1}{R}\frac{dR}{dv}\right)_0$	−0·206	−0·256	−0·26	−0·29	Mean = −0·254

At the lower range of voltage the characteristic is evidently of the type II. of Fig. 5.

Iron-Nickel.—The characteristic is of the same type as the last. The voltage-coefficient found was −0·16, and a minimum of resistance occurred at v_m=0·016 volt. The droops occur later than in the preceding case, and no difficulty was found in obtaining the point of minimum.

Constantan-Manganin.—This constitutes the case b=0. The characteristic equation (7) reduces to $R=F_0'-\frac{a}{K}v$, which represents a straight line. This linear type of characteristic was found as expected. An example is shown in Fig. 10; the duration of the applied voltage was 0·1 second.

The values of the voltage-coefficient for five different contacts are as follows :—

TABLE VII.—*Constantan-Manganin Contacts.*

No.	1.	2.	3.	4.	5.	—
R_0	0·0735	0·108	0·112	0·171	1·30	
$(dR/dv)_0$	−0·0083	−0·0095	−0·012	−0·0195	−0·133	
$\left(\frac{1}{R}\frac{dR}{dv}\right)$	−0·113	−0·088	−0·107	−0·114	−0·103	Mean = −0·105

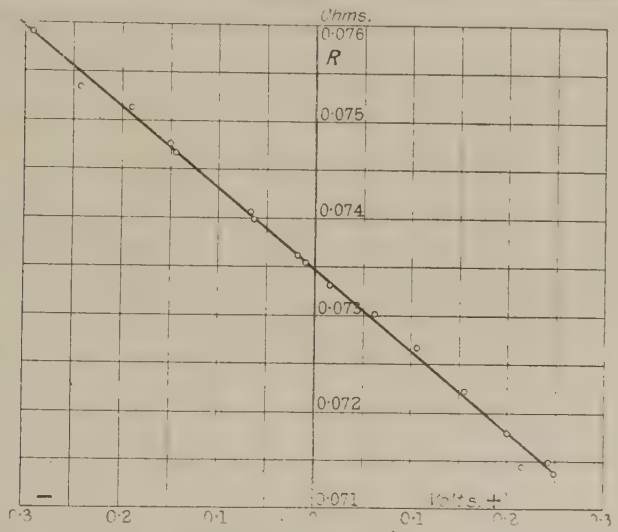

FIG. 7.—A CONSTANTAN-MANGANIN RESISTANCE CHARACTERISTIC.

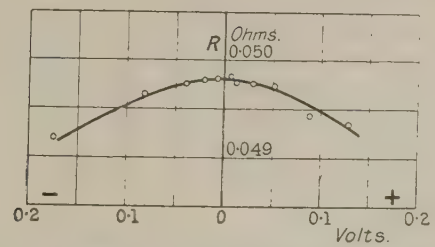

FIG. 8.—A COPPER-MANGANIN RESISTANCE CHARACTERISTIC.

Copper-Manganin.—Here $a \doteqdot 0$, $b \doteqdot 0$, so that on the theory R should be independent of v. Fig. 8 represents a specimen set of observations. The curve is symmetrical about the

vertical axis. The slight fall is attributed to the softening effect.

A similar curve, though still flatter, was found with carbon-manganin.

Bismuth-Bismuth.—The object of this test was to examine the view of a softening due to rise of temperature. The characteristic, shown in Fig. 9, possesses a slight asymmetry due possibly to a slight difference in chemical quality of the two components of the contact. The anticipated droops were found to occur very sharply. If the duration of the current is prolonged the resistance goes on falling, and past a certain point the original resistance cannot be reproduced. The effect is paralleled exactly in the case of mineral contacts at the higher voltages.

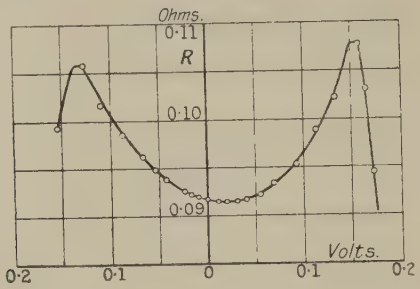

Fig. 9.—A Bismuth-Bismuth Resistance Characteristic.

V. *Some Calculations.*

The Value of KR_0.—When the values of a and b are known this can be calculated from equation (10). For antimony-bismuth the value is $8{\cdot}6\times10^{-4}$; for iron-nickel $4{\cdot}3\times10^{-4}$; and for manganin-constantan $3{\cdot}9\times10^{-4}$. These values are all of the same order, thus supporting the views of the nature of the contact proposed in Section IV., according to which the value of this quantity is determined by the ratio of the thermal and electrical conductivities, which is practically a constant in the case of metals.

The Ratio R/R'.—By equation (7) the ratio of the apparent and real resistance of a contact at zero voltage is given by the expression $(1+a^2T_0/KR_0')$. Applying the calculation to the case manganin-constantan, we have $a=40\times10^{-6}$, $KR_0=3{\cdot}9\times10^{-4}$, whence $a^2T_0/KR_0'=0{\cdot}0012$, which is very small compared

with unity. The same holds practically in the case bismuth-antimony.

The Rise of Temperature at the Contact.—Substituting for R the value $R_0'v(1+bt)/(v+at)$ derived from equations (1), (3) and (4), we obtain an expression for t in terms of v, which for small values of v reduces to

$$t=-v\,.\,\frac{aT_0}{KR_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The temperature change is thus proportional to the applied voltage and to the thermo-electric power. In the case of constantan-manganin the equation becomes $t=-30{\cdot}5v$, which denotes a fall of temperature for current in the direction constantan to manganin across the contact, at the rate of

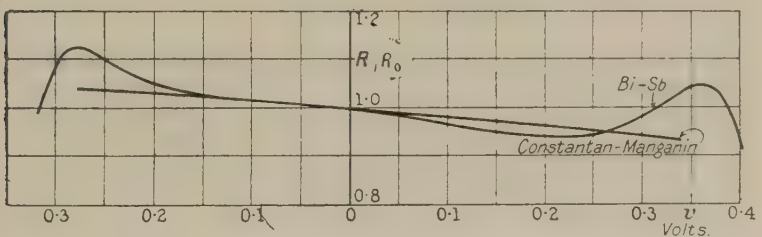

Fig. 10.—The Specific Resistance-Characteristics of two Metallic Contacts.

30·5°C. per volt. For antimony-bismuth $t=-36v$. We may obtain an estimate in the case of mineral contacts by assuming values for (a) and (b). Thus, in the case of galena we may accept roughly $a=500\times10^{-6}$, $b=-0{\cdot}01$, whence $t=-177v$, *i.e.*, a fall at the rate of 177°C. per volt for current crossing the contact from galena to platinum.

The Value of v_m.—The agreement between the observed values of v_m and those calculated by equation (11) are shown in the Table below :—

Contact.	$a\times10^6$	b	R_m/R	v_m	
				Calculated.	Observed.
Iron-nickel	32	+0·004	0·998	0·008	0·016
Antimony-bismuth	100	+0·004	0·96	0·027	0·22
Galena-platinum ...	500 (assumed)	−0·01 (assumed)	1·04	0·05	0·066
Zincite-platinum ...	1,000 (assumed)	−0·01 (assumed)	1·54	0·136	0·15

VI. *The Reception of Electric Waves by the Contact Rectifying Detector. Conditions of Highest Sensibility.*

The law of constancy of the voltage coefficient can be applied to the receiving circuit in which a contact detector is employed in series with a magnetic telephone. It will be supposed, in the first place, that no polarising voltage is used. Since the oscillatory E.M.F. required to produce a signal is only a small fraction of a volt, the relation between R and v is practically a linear one. Let

$v=v_0 \sin pt$=applied voltage.
i=instantaneous current.
R=apparent resistance of contact.
R_t=resistance of telephone.
$A=\left(\frac{1}{R}\frac{dR}{dv}\right)_0$.

A simple treatment will be adopted, the induction of the telephone and leads being neglected and the resistance assumed to be independent of frequency. The conclusions thus reached may be corrected in the direction required by the practical conditions applying in any given case.

The current at any instant is given by

$$i=\frac{v_0 \sin pt}{R+R_t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also $$R=R_0(1+Av) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence $$i=\frac{v_0 \sin pt}{R_t+R_0+R_0 A v_0 \sin pt}, \quad . \quad . \quad . \quad (22a)$$

which, since v_0 is considered small compared with unity, may be written with sufficient accuracy

$$i=\frac{v_0 \sin pt\left(1-\frac{AR_0}{R+R_0} \cdot v_0 \sin pt\right)\sin pt}{R_t+R_0} \quad . \quad . \quad (22)$$

The mean or rectified current is

$$\frac{1}{T}\int i dt=\frac{-AR_0 v_0^2}{2(R_t+R_0)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Its value is proportional to the square of the R.M.S. value of the applied voltage, a result first empirically deduced by L. W. Austin.*

* Austin, "Phys. Rev.," 1907.

The pull on the telephone diaphragm is determined by the number of ampere-turns. The best resistances of contact and telephone are those which make the ampere-turns a maximum for a given applied voltage. For maximum sensibility of receiving circuit, therefore, $\frac{nAR_0}{2(R_t+R_0)^2}$ must be a maximum, where n denotes turns on telephone.

Three cases will be discussed :—

i. The telephone supposed given, *i.e.*, n and R_t fixed. The ampere-turns are then proportional to $\frac{AR_0}{(R_t+R_0)^2}$, which is a maximum when $R_0=R_t$.

ii. The contact resistance supposed given. Assuming a constant weight of copper in the coil, and neglecting the correction for space occupied by insulation, we have $R=kn^2$. The required condition then reduces to $\frac{\sqrt{R_t}}{(R+R_0)^2}$=maximum. The solution is $R_t=\frac{1}{3}R_0$.

iii. The contact resistance and resistance of telephone variable at choice. The condition now is $\frac{R_0\sqrt{R_t}}{(R_t+R_0)^2}$=maximum. As R_0 is varied it is necessary, as the preceding case shows, to follow suit by rewinding the telephone so that $R_t=\frac{1}{3}R_0$. The condition thus reduces to $R_0^{-\frac{1}{2}}$=maximum. The smaller the resistance of contact and telephone the better. In practice, however, a limit to this reduction is evidently imposed by the impedance in other parts of the receiving circuit, which has not been considered.

To sum up, we conclude that for high sensibility :—

(1) The contact should be of materials having as large a voltage-coefficient as possible ;

(2) The resistance of the contact should be low (the lowest value depending on the impedance in receiving antenna, &c.) ;

(3) The resistance of the telephone should be about one-third that of the contact.

Use of a Polarising Voltage.—By applying the above considerations, and paying due regard to the form of the resistance characteristic for any combination, the effect of a polarising voltage may be foreseen. It is evident that if the polarising voltage be such as to lead to the point of maximum resistance on the characteristic, then the rectifying power and the sensibility become zero.

Again, if the resistance of the contact is below that of the telephone, it is quite possible that a polarising voltage in either direction may be deleterious or at least ineffective.

It is when the resistance of the contact happens to be much larger than that of the telephone that the advantage of a polarising voltage will be decisively seen. Its sign must evidently be chosen in the direction of the initial fall in the characteristic, *e.g.*, for the perikon the polarising current must cross the contact from chalcopyrites to zincite.

An examination of the curves shows that with voltages increasing from zero in the direction just specified, the slope of the characteristic at first increases. As the polarising voltage is increased further the slope at the corresponding points of the characteristic falls off, and evidently a value of the polarising voltage will be reached at which a maximum effect occurs. This value will depend on the initial resistance of the contact; it is thus not possible to specify any definite figure for the optimum polarising voltage at a contact of given materials.

Theoretically it appears that, given the receiving telephone, the ideal contact is one to which is applied the small polarising voltage leading to the point of maximum slope on the characteristic; and of initial resistance such that under that polarising voltage the resistance falls to a value exactly equal to that of the telephone.

VII. *Use of Direct-Current Galvanometer in Series with a Rectifying Contact in Null Alternating-Current Bridge Methods.*

This problem is amenable to the treatment applied in the last section. If the galvanometer be regarded as given, then for highest sensibility the resistance of the contact should be equal to that of the galvanometer, but if a given contact be taken then the best galvanometer to choose is one of resistance one-third that of the contact at zero voltage.

Treated as a millivoltmeter, we see that the arrangement shares with hot-wire instruments and dynamometers the defect that the square law is involved; and it may be shown that the minimum voltage detectable cannot be reduced greatly below a millivolt. For currents within the range of useful frequency of the telephone and the vibration galvanometer the combination of contact and direct-current galvano-

meter thus falls much below those instruments in sensibility. This conclusion is in accordance with experimental results.

The following example of a test may be quoted. The agreement between the observed rectified current and that calculated by equation (23) is very satisfactory :

Contact of galena-platinum : $A=1{\cdot}20$.

$R_0=95$ ohms.

Galvanometer resistance $=250$ ohms.

Sensibility $=186$ scale-divisions (1/40 in.) per microampere.

Applied voltage (R.M.S.)$=0{\cdot}05$ volt, at 85 $\sim$.

Observed steady deflection $=383$ scale-divisions ; whence rectified current $=2{\cdot}06\times10^{-6}$ amperes.

Rectified current calculated by (23)

$$=\frac{1{\cdot}2\times95\times0{\cdot}05^2}{345^2}=2{\cdot}38\times10^{-6}\text{ amperes.}$$

Taking half a scale-division as the minimum observable deflection, the minimum alternating voltage detectable with the given galvanometer and contact $=\frac{0{\cdot}05}{\sqrt{766}}=1{\cdot}81\times10^{-3}$ volt.

Summary.

1. Measurements by the Wheatstone bridge have been made of the electric resistance at various rectifying mineral contacts ; the results being expressed in the form of resistance characteristics. The thermo-electric sign of the current was found in each case.

A series of similar measurements are recorded on contacts both elements of which are metallic.

2. A formula for the resistance characteristic is derived on the view that the phenomena in question are due to thermo-electric action. The experimental results are shown to be fully explained on this view. The fact that the rectified current (when an alternating voltage is applied) is generally, in the case of mineral detectors, in the opposite direction to that produced by heating the contact, is shown to be in harmony with the thermo-electric view, and not, as has generally been supposed, in collision with it. The determining factor in respect of sign proves to be a large negative temperature coefficient of resistance.

3. In relation to the thermo-electric theory, two views as to the location of the resistance at a contact are proposed and discussed.

4. Some calculations, including that of the rise of temperature at a contact, are given.

5. The experimental results form the basis of calculation of the best values to chose for the resistances of the contact and telephone in a wireless receiving circuit.

6. The influence of a polarising voltage on the sensibility of the receiving circuit is discussed.

7. It is shown that the combination of direct-current galvanometer and sensitive contact cannot respond to an alternating voltage much below 1 millivolt.

8. Observed values of rectified current are found to be in close agreement with those calculated on the thermo-electric theory.

ABSTRACT.

The Paper contains an account of an investigation, the primary object of which was to determine the nature of the physical actions occurring at a rectifying contact. Resistance characteristics are given for various contacts, some including a mineral, some in which both elements are metals. It is shown that a *specific characteristic* may be drawn for any given pair of materials. The experimental results are in accordance with the view that the actions are thermo-electric, the main determining factors being the thermo-electric power and the temperature-coefficient of electric resistance.

Based on the law of constancy of the voltage-coefficient calculations are given showing the best value of the resistance of the telephone in a wireless receiving circuit in which the contact detector is employed. The influence of a polarising voltage is also traced.

The use of the combination of rectifier with a direct-current galvanometer as indicator of the balance point in an alternating-current bridge is examined, and it is shown that the minimum detectable alternating voltage cannot be reduced much below a millivolt.

DISCUSSION.

Mr. DUDDELL thought the Paper was a very important contribution. He had no definite opinion as to the true explanation of rectifying contacts. In practice very much depends on the exact part of the crystal at which contact is made. With regard to thermo-electric forces, he had examined a large number of crystals in the hope of obtaining contacts which were more stable than those ordinarily employed, and he had concluded that no mineral which did not show thermo-electric effects was of any use as a detector. There might be exceptions in the case of metals, but the rule applied to all sulphides. His experience with regard to the receiving telephone differed somewhat from the author's. In a telephone the resistance and impedance depend on the frequency. In practice it seemed desirable to keep the impedance of the telephone of the same order as the resistance of the detector, and adjust the latter until the desired sensitivity is attained. In receiving strong signals one may employ very tight contacts in which it appears to be impossible that there should be any air film. A satisfactory rectifying action is still, however, obtained.

Prof. G. W. O. Howe thought that it was well to avoid flying off to new hypotheses in order to explain new phenomena if they could possibly be accounted for on well-known grounds. He was pleased to see that the author had been able to explain rectifying contacts without invoking any mysterious new properties. He was rather surprised to gather from the author's remarks that the thermo-electric theory was generally discredited. His impression had been that the work of Dr. Eccles had established it as the accepted explanation. Had any of the exponents of the electron theory attempted to give a detailed explanation of the observed phenomena on that basis? What was the origin of the "settling down" of contacts mentioned by the author? Was it a mechanical or thermal effect? He would like to hear just in what respect the results of Mr. Owen's work differed from those of Dr. Eccles.

Dr. W. Eccles remarked that he was glad to find such agreement on the absurdity of rushing to special explanations of these phenomena. Certain workers had thought that electrostatic attraction altered the area of contact. Others assumed the opening of trapdoors to let electrons through, the mechanism of the process being considerably more difficult to explain than the original results themselves. Prof. Fleming had found that many of the substances used as detectors were photo-electric—*i.e.*, they give off electrons under the action of light—and he assumed that rectifying phenomena were caused by a similar action of the longer electro-magnetic waves. He (Dr. Eccles) had tried to put these points straight by explaining the phenomena in terms of well-known physical properties, such as the resistance coefficient, Peltier effect, &c., and had found it possible to account for all the effects. The theory proved unpopular because it had nothing occult about it. Mr. Owen had dropped the Thomson effect from his treatment. He thought this was not justifiable, as the Thomson effect was a physical fact which must enter into the result in practice. Leaving it out of consideration simplifies the treatment, but loses the explanation of several of the erratic properties of detectors. For instance, the theoretical resistance characteristic curves obtained by Mr. Owen are parabolic. It is obvious, however, that the experimental curves are of the fourth degree. The Thomson effect must be introduced to explain the way the curve bends off at the extremities. One important result of the author's was that for the same materials the value of R_m/R_0 was constant for any contact, light or heavy. This appeared to him to be due to the fact that increasing the area of contact by increasing the pressure was equivalent to putting two or more similar contacts in parallel.

Mr. A. F. Hallimond (communicated): I should like to congratulate the author on his reduction of the curves for each contact-pair to a specific resistance characteristic. If it is found that the same process can be employed in all cases a very useful and simple means of comparison will be established. The result seems the more remarkable in being based upon the resistance-curves, and I should like to ask whether the relation throws any light on the conditions obtaining at the contact when the pressure is varied. With reference to the heating of the contact, I believe that Prof. Pierce observed that at 80 cycles no lag occurred, such as might be due to heating. The author's arrangement by which the current may pass from 0·003 second would seem to be even more sensitive. Was any variation observed in the resistance with the time of passage of the current? With regard to the thermal effects, might I suggest the investigation of pyrite, a mineral in which adjoining areas may exhibit widely different thermal E.M.F.s?

Mr. P. R. Coursey communicated the following remarks: The three main results arrived at will be of especial value, even if they do no more than form a basis of comparison for the work of different investigators, which up to the present has only been applicable (quantitatively) to the particular

contacts on which the experiments were made. I should, therefore, like to ask the author whether he has found the "specific resistance characteristics" that he has obtained to be applicable to results derived from measurements on different crystals of the same materials, or merely to contacts of various pressures and resistances between individual crystals. The two very different curves given for carborundum crystals would seem rather to support the latter view. As to the accuracy of this method of determining the characteristic, it would not seem to possess much ultimate advantage where very small voltages are concerned, as in any case the limit of accuracy depends on the galvanometer employed, whether for the direct measurement of the current passing through the crystal or for the indication of balance of the bridge. Moreover, the method of obtaining the resistance with very brief currents, although leading to consistent results from the "curve" point of view, scarcely corresponds to the use in practice of a steady polarising or boosting voltage (often of the order of 1 volt, as is usual, for instance, with most carborundum crystals) which may be impressed on the crystal for many hours at a time, and would, therefore, probably produce a maximum of softening effects at the contact. It would, therefore, be interesting if a series of curves could be obtained with different contact times in order that the magnitude of this effect might be gauged. On the other hand, it is just possible that if any appreciable temperature rise occurs at the contact this factor of itself may cause an increase in the sensitiveness of the detector, quite apart from the purely electrical effect of the applied voltage. These considerations, however, perhaps affect rather the practical applications of the detectors than the physical processes underlying their action with which the Paper is chiefly concerned.

The Author (in reply) said he was glad to learn that it was Mr. Duddell's uniform experience that high thermo-electric power and sensitiveness of a contact detector go together. In regard to the relation of impedance of telephone to the resistance of the contact he was quite prepared to find that the conclusions deduced from the simple treatment employed in the Paper required modification to conform to the actual conditions of practice. In reply to Prof. Howe's inquiry as to the cause of the "settling down," this was not due to thermal action; the small testing voltage was applied only at intervals, and was not itself the cause of the change described. The effect appeared to be of the nature of an adjustment of the molecules bounding the contact when their setting was changed; it was probably allied to the slow changes met with in the study of elastic strains. He did not go so far as to say that the thermo-electric theory had become generally discredited, but rather that it stood in need of confirmation. He would endeavour briefly to point out in what relation his own work appeared to stand to that of Dr. Eccles; this reply would cover some points of difference which Dr. Eccles had commented upon. Dr. Eccles employed coefficients representing the Peltier effect, the Thomson effect, and the temperature coefficient of resistance. The author finds that the existence of softening and consequent increase of area of contact due to rise of temperature becomes very important at the higher voltages. This effect is complicated and thus not amenable to mathematical treatment; the course adopted in the present Paper is, therefore, to restrict the application of the formulæ within small limits of voltage—say, a few tenths of a volt. Apart from this, the chief point of difference is in the emphasis laid on the relative value of the coefficients. The author finds that the experimental results in the case of all the mineral and metallic contacts tested may be explained without invoking the Thomson effect: the conclusion is, therefore, drawn that this term is of small importance in comparison with the Peltier effect, at all events at low voltages. It thus fortunately becomes possible to simplify the theory appreciably. Again, in the author's view not only does resistance admit of more precise measure-

ment, but the statement of results in the form of resistance characteristics brings out the essential features of the behaviour of these contacts more clearly than if current characteristics are employed. In reply to Mr. Hallimond, the determination of resistance of contact as a function of pressure is not specially considered in the Paper. The tests here recorded tend, however, to show that pressure only enters in so far as it alters the area of contact. In regard to the question of time-lag, this is the subject of experiments now in progress. In reference to pyrite, the author fully grants that heterogeneity may and does exist in this and other cases; for further instance, in the high and low-resistance specimens of carborundum referred to in the Paper, both possessing high thermo-electric power, but of opposite signs in the two cases.

In reply to Mr. Coursey's first question, it is a matter for experiment to discover to what extent various specimens of nominally the same mineral agree in behaviour. Reference may be made to the remarks immediately preceding. As to the matter of accuracy, the bridge method carries the advantage that a null method has over a deflectional one; and, further, that of permitting of practical elimination of the effects caused by prolongation of time of application of applied voltage. It is, of course, true that when a polarising voltage is applied the time effect must enter; but even here the all-important changes caused by the superposition of a rapidly oscillatory voltage are in themselves independent of softening action.

XIX.—*The Latent Heat of Fusion of a Metal and the Quantum-Theory.* *By* H. Stanley Allen, *M.A., D.Sc., University of London, King's College.*

§1. When a pure crystalline solid is heated, fusion takes place over the surface only at a definite temperature, T. The atomic heat of fusion of the solid is the number of calories required to melt 1 gram atom at this temperature, *i.e.*, it is AL, where A is the atomic weight and L is the latent heat per gram.

The energy supplied to melt the solid may be identified with the work that must be done to break the bonds which impart to the solid its crystalline structure. The molecules liberated by the breaking of the bonds are, in general, free to move amongst their neighbours, so giving the mobility of the liquid state.

According to the quantum-theory, the energy of a solid is the energy of the oscillators present. From this standpoint it would appear that the latent heat of fusion may be identified with the energy necessary to counterbalance that of a certain number of these oscillators. In this theory the average amount of energy that is associated, at absolute temperature T, with a vibration of frequency ν is supposed to be

$$RT\times\frac{x}{e^x-1},$$

where x stands for $h\nu/RT$, h is Planck's constant and R is the gas constant for a single molecule. This expression is employed both by Einstein and by Debye in discussing the theory of the specific heat of a solid.*

Let c denote the ratio of the number of oscillators concerned in holding together the crystalline structure to the number of atoms, then the number of these oscillators in 1 gram molecule is Nc where N is Avogadro's constant. We now assume as the fundamental equation for the latent heat of fusion of a monatomic metal

$$AL=cNRT\times\frac{x}{e^x-1}. \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

Hence

$$\frac{AL}{NRT}=c\frac{x}{e^x-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

* Jeans, "Report on Radiation and the Quantum-Theory," Ch. VI., 1914.

Here NR is the value of the gas constant for 1 gram molecule, which is known to be 1·989 calories per gram molecule per degree.

To be able to test the validity of the hypothesis it is necessary to know the value to be assigned to ν, the frequency of the oscillator concerned in the change of state, at the temperature of the melting point. For this purpose the formula of Lindemann* has been employed. This gives

$$\nu = k\sqrt{\frac{T}{AV^{\frac{2}{3}}}}, \quad \ldots\ldots\ldots \quad \text{(iii.)}$$

where V is the atomic volume and k is a multiplying factor, whose value, deduced empirically from the case of silver, is $2{\cdot}12\times10^{12}$. The formula was found by Lindemann to give results in fair agreement with those determined from the atomic heats by Einstein's formula in the case of the elements Pb, Cu, I, Pt and Si.

TABLE I.—*Experimental Constants for the Metals.*

Metal.	A.	L.	T.	V.	ν.
Aluminium	27·1	76·8	931	10·1	$5{\cdot}75\times10^{12}$
Bismuth.....	208·0	12·64	541	21·2	1·24
Cadmium	112·4	13·66	593·7	13·0	2·07
Cæsium	132·81	3·766	301·4	71·0	0·77
Copper	63·57	42·0	1,356	7·1	*4·93
Gallium	69·9	19·0	303	11·79	1·94
Iron	55·84	59·0	1,803	7·1	6·27
Lead	207·1	5·36	600	18·2	*1·44
Mercury	200·6	2·82	234	14·8	9·33
Nickel	58·68	4·64	1,708	6·7	6·07
Palladium	106·7	36·3	1,818	9·3	4·16
Platinum	195·2	27·2	2,028	9·1	*3·1
Potassium	39·1	14·67	336·5	45·5	1·74
Rubidium	85·45	6·144	312	56·0	1·06
Silver	107·88	21·07	1,234	10·3	*3·3
Sodium	23·0	27·21	370·9	23·7	2·96
Tin	119·0	14·0	505	16·2	1·73
Zinc	65·37	28·13	692	9·5	3·26

* Lindemann.

§2. The data available for the latent heats of the metals are limited, and there can be no doubt that the values in many cases are subject to a large percentage error. The values for L and T in Table I. are taken from the Smithsonian Physical Tables (1914) and " Recueil des Constantes Physiques " (1913), with the exception of those for the alkali metals, which have

* Lindemann, " Phys. Zeitschr.," Vol. XI., p. 609, 1910.

been investigated recently by Rengade.* The temperature of the melting point is given on the absolute scale. The values for the atomic volumes are those given by Richards.†

§ 3. In Table II. are given the values of the change in entropy, AL/T, and of AL/NRT, arranged in order of magnitude. The calculated values for the latter quantity were obtained by assigning to the factor c a value which was either unity or a simple fraction, as shown in the last column of the Table. It must be noticed that the value assigned to c is

TABLE II.—*Atomic Latent Heat of the Metals.*

Metal.	AL/T.	AL/NRT.		
		Obsd.	*Calcd.*	*c.*
Nickel	(0·159)	(0·080)	0·915	1
Caesium	1·66	0·835	0·939	1
Rubidium	1·68	0·845	0·919	1
Sodium	1·69	0·850	0·809	1
Potassium	1·70	0·855	0·878	1
Iron	1·827	0·919	0·918	1
Silver	1·842	0·926	0·915	1
Lead	1·850	0·930	0·942	1
Copper	1·969	0·990	0·915	1
Palladium	2·130	1·07	0·946	1
Aluminium	2·236	1·12	1·143	4/3
Mercury	2·418	1·22	1·208	4/3
Cadmium	2·586	1·30	1·221	4/3
Platinum	2·618	1·32	1·289	4/3
Zinc	2·657	1·34	1·338	3/2
Tin	3·299	1·66	1·609	7/4
Gallium	4·382	2·20	2·182	5/2
Bismuth	4·860	2·44	2·363	5/2

based on the assumption that the element is monatomic. If this be not the case, Lindemann's formula and the value of c may require modification.

A comparison between the calculated and the observed values of AL/NRT shows moderately good agreement, perhaps as much as we have any right to expect in view of the uncertainty attaching to some of the quantities involved. In the case of the alkali metals it will be noticed that the order for the calculated values is that of the periodic table. The observed values for

* Rengade, "C.R.," Vol. CLVI., p, 1897, 1913.
† Richards, Am. Chem. Soc. "Journ.," Vol. XXXVII. p. 1643, 1915.

rubidium and cæsium are probably under-estimates. It is clear that the value recorded for the latent heat of nickel (4·64 calories per gram, Pionchon, 1887) is not appropriate for a pure crystalline solid of melting point 1,708°K. It is more likely to represent the value for a mixture of crystalline and amorphous material. The well-known modifications in the magnetic properties of nickel resulting from temperature changes are in agreement with such a view.

§ 4. Oxley* has pointed out that intense local molecular fields are necessary to account for the rigidity of a crystalline structure. If the field be interpreted as a magnetic field the strength must be of the order 10^7 gausses, and "the amount of potential energy associated with 1 gm. of the crystalline medium in virtue of its molecular grouping" must be about 10^9 ergs. "The thermal equivalent of this will be of the order $10^9/4\cdot10^7=25$ gm. calories, which represents the heat energy required to destroy the crystalline structure—*i.e.*, the latent heat of fusion."

Without discussing the particular interpretation of the molecular field put forward in Oxley's Papers, it will be seen that his view of the latent heat of fusion is not inconsistent with that here adopted.

In 1897 Prof. Richards† concluded that the atomic heat of fusion of a metal is usually $\frac{1}{3}$ (or $\frac{1}{2}$ or 1) the heat required to raise the temperature of the metal from absolute zero to the melting point T, or, assuming the law of Dulong and Petit, AL is approximately $2\cdot1\ T$. The figures recorded in Table II. do not lend much support to this view.

Attempts have been made to find expressions for the latent heat of fusion of compounds, and some of these are of interest in connection with the present discussion.

Crompton‡ in 1895 suggested that the latent heat of fusion could be calculated from the formula,

$$ML/T\Sigma v = \text{a constant } (1\cdot38),$$

where M is the molecular weight and Σv the sum of the valencies. There is a certain correspondence between this formula and equation (ii.) of the present Paper, for the latter may be re-written as

$$AL/Tc=NR\frac{x}{e^x-1}.$$

* Oxley, "Phil. Trans.," Roy. Soc. A, Vol. CCXV., p. 93, 1914.
† Richards, "Chem. News," Vol. LXXV., p. 278, 1897.
‡ Crompton, "Journ." Chem. Soc., Vol. LXVII., p. 315, 1895.

Since $\frac{x}{e^x-1}$ in the cases considered is not far from unity, the right-hand side of this equation is very approximately constant.

Walden,* from analogy with the Pictet-Trouton rule for the latent heat of vaporisation, gave the relationship ML/T =constant. According to Tamman,† the latter rule is of doubtful validity, as only about one-quarter of the substances examined gave values for the constant falling within the limits 15·8 to 12·0 suggested by Walden. "Metals give constants ranging from Bi 4·82 to K 1·84."

§ 5. *Application of Debye's Theory.*—The form of quantum-theory employed above is that due to Einstein, in which all the vibrations in a solid at a specified temperature are assumed to have the same frequency, ν. It is known that this theory is imperfect, and must be regarded as at best a first approximation to the truth. In the theory of Debye‡ the frequencies are dependent on the elastic constants of the solid, and may assume all values between zero and a certain maximum value, ν_m. "His theory is based on a number of hypotheses which cannot be accepted as being strictly valid, but of the formulæ hitherto published for the representation of the atomic heat curves of the metals that of Debye gives the closest approximation to the experimental results."§

I have attempted to apply this theory to the determination of the latent heat of fusion by finding the energy in a substance at the melting point both in the state of solid and in the state of liquid. Unfortunately, the only metal for which the compressibility has been determined when liquid is mercury. In this case the theory does not give a result in harmony with experience. The result is improved somewhat by the introduction of a zero point energy, as in the later form of Planck's theory. If the energy of an oscillator be taken as

$$\frac{h\nu}{e^{h\nu/kT}-1}+\alpha h\nu,$$

close agreement between the observed and the calculated values of the latent heat can be obtained by assuming $\alpha=\frac{4}{5}$, instead of the usual assumption $\alpha=\frac{1}{2}$. This assumption is,

* Walden, "Zeitschr. Elektrochem.," Vol. XIV., p. 713, 1908.
† Tamman, "Gese l. Wiss. Göttingen," Vol. III., p. 335, 1913.
‡ Debye, "Ann. de Physik," Vol. XXXIX., p. 789, 1912.
§ E. H. Griffiths and E. Griffiths, "Proc." Roy. Soc., A, Vol. XC., p. 559, 1914.

however, entirely arbitrary, and there is no theoretical ground for adopting it. Determinations of the compressibility of other metals in the liquid state are much to be desired.

§ 6. To sum up the results of the present Paper, it may be said that the atomic heat of fusion of a metal can be calculated with fair accuracy on the basis of the quantum-theory by assuming that the latent heat represents the energy of a certain number of oscillators having a frequency at the melting point which can be found from the formula of Lindemann.

ABSTRACT.

The latent heat of fusion is identified with the energy necessary to counterbalance that of a certain number of "oscillators" concerned in holding together the crystalline structure. Assuming that the energy of an oscillator having a vibration frequency ν is

$$RT \times \frac{x}{e^x - 1},$$

where x stands for $h\nu/RT$, it is found that the atomic heat of fusion of a metal can be calculated with fair accuracy by the formula,

$$AL = cNRT \times \frac{x}{e^x - 1}.$$

Here A denotes the atomic weight, L the latent heat, and c the ratio of the number of oscillators in question to the number of atoms. Thus, the number of oscillators in 1 gramme molecule is Nc, where N is Avogadro's constant. It is found that to the factor c must be assigned a value which is either unity or a simple fraction. The frequency at the temperature of the melting point is calculated by means of the formula of Lindemann. The application of Debye's theory is also discussed.

DISCUSSION.

Dr. R. S. Willows agreed with Dr. Allen that the quantum-theory was still in a very speculative state. He referred to the statement in § 1, that "*The energy supplied to melt the solid may be identified with the work that must be done to break the bonds which impart to the solid its crystalline structure.*" This seemed to rule out of consideration the case of amorphous solids. It is supposed that some of the alterations produced in a metal by treatment such as rolling are due to some of the molecules becoming amorphous. Could the theory be extended to deal with the change from the crystalline to amorphous states?

Dr. Allen thought the change from crystalline to amorphous conditions was the same as from crystal to liquid.

Prof. Boys: Would not that imply that there was no latent heat in amorphous material?

Dr. Allen agreed that that would be implied.

Mr. Ezer Griffiths communicated the following remarks: Could the author give a clearer conception of the theory advanced to explain the process of liquefaction on the quantum-theory, and in particular of the statement "that the latent heat of fusion may be identified with the energy necessary to counterbalance that of a certain number of these oscillators?" Was this statement in harmony with that postulated by Lindemann in

obtaining the frequency ν—namely, "that the melting point is determined by the fact that at this temperature the amplitudes of the vibrations of the atoms around their position of rest become commensurable with the atomic distances—*i.e.*, it is the collision of the atoms which produce the destruction of the crystalline form?" Further, the statement that "the energy of a solid is the energy of the oscillators present" needed some qualification, particularly at high temperatures. Sodium, for example, had an atomic heat just below the melting point 7·5, while the calculated value was only about 6·5.

Communicated reply to Mr. Ezer Griffiths: I have refrained of set purpose from attempting to give an exact description of the process of fusion, or of the nature of an "oscillator." It is characteristic of the quantum-theory that it is capable of interpretation in various ways, and different mechanisms might, no doubt, be imagined which would satisfy the requirements of the theory.

Personally I regard Lindemann's formula as largely empirical, but I do not think the theory of my Paper is inconsistent with the view of Lindemann, if that be expressed by saying that the destruction of the crystalline form is conditioned by (not necessarily produced by) the collision of the atoms.

It is known that at high temperatures complications may arise—*e.g.*, through the rotations of the atoms or molecules (Jean's Report, pp. 76, 77), but I have not attempted to deal with these at the present stage of the work.

XX. *Lenses for Light Distribution.* *By* T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received April 1, 1916.

The conditions to be satisfied by the lenses of motor car headlights, ship's sidelights and headlights and a number of other lenses used to illuminate near or distant objects or to indicate the presence or position of a ship or other body are so dissimilar from those involved in what are ordinarily included among optical instruments that these lenses cannot be satisfactorily designed on the principles successfully employed in determining the forms for the lenses of the more usual class of optical instruments. A method which is applicable to the kinds of lenses enumerated above will be apparent from the discussion of a special case.

The starboard and port lights of steamers are required to be visible over a definite extended horizontal angle from as great a distance as possible. Owing to the rolling of the ship it will not be satisfactory to project the light with a very small vertical divergence from the horizontal plane. On the other hand, the visibility required is such that with the available sources of light and the desired size of lantern the necessary horizontal intensity cannot be secured over the whole vertical angle which it is desired to illuminate.

In connection with an investigation by Paterson and Dudding on the visibility of ships' navigation lights it was desired to obtain uniform intensity in the projected beam over the angle between 5 deg. of elevation and 5 deg. of depression, and to let the intensity fall off uniformly outside these limits at such a rate that when the lantern was inclined at 15 deg. to the horizontal the visibility at a considerable distance would be equal to that of the central beam at twice that distance.

Fig. 1 gives the relation between the intensity and the angle of elevation or depression when half visibility is required at 15 deg.

The same light distribution is required for all azimuths, and the lenses are, therefore, cylindrical with the source of light on the axis of the cylinder. If this source may, with sufficient accuracy, be considered a line source coincident with the axis of the cylinder the problem becomes two dimensional.

It has been customary in the past to make the inner surfaces of these lenses in the form of a right circular cylinder, and this

form is adopted in the present case. The general dimensions of the lamps now in use, in which the distance of the inner glass surface from the vertical axis is from 4 in. to $4\frac{1}{2}$ in. and the vertical aperture of the lenses from 6 in. to 7 in. are also adhered to. These dimensions have a very appreciable influence on the shape of the lens, since the ratios of these lengths to the height of the source of light are some of the most important among the factors which determine that shape. When electric glow lamps are used this height may be easily adjusted to suit the dimensions of the lantern, but as it is necessary for the same lanterns to be suitable for use with lamps burning colza or paraffin, the height of the source of light to be used is fixed within comparatively narrow limits.

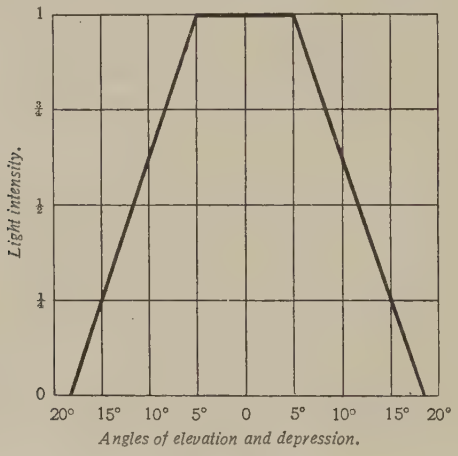

Fig. 1.

There are two distinct cases that may arise : in the one the decline in the light intensity beyond 5 deg. is due to the eclipse of part of the light source by the lens mount, total eclipse taking place at the extreme angle at which any light is emitted ; in the other, the falling off is due to a change in the curvature of the outer lens surface. In general, the former type requires a high light source, the latter a comparatively short one. In both types the internal principal focus of the lens must be further from the lens than the axis. The mathematical treatment of the former case is not simple, and will not be given here, as the conditions rarely apply to practical cases.

Suppose, then, that a small linear source of light is coincident

with the axis of the system which is assumed to be symmetrical with respect to the horizontal plane bisecting the source.

Let a ray of light start from the source making an angle θ with the horizontal until it meets the inner glass surface of radius a. After refraction let it make an angle φ with the horizontal, and be refracted from the lens at an angle β, the tangent to the refracting surface making an angle α with the vertical. Then, if μ is the refractive index of the glass

$$\sin \theta = \mu \sin \varphi,$$

and

$$\mu \sin (\alpha - \varphi) = \sin (\alpha - \beta).$$

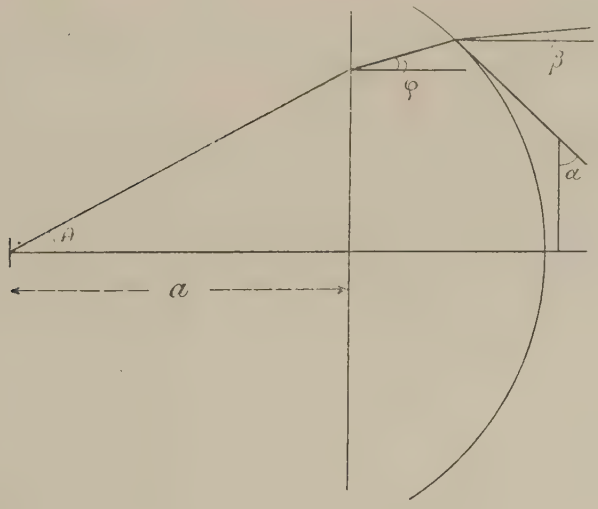

FIG. 2.

Denote by β_1 the angle (in this case 5 deg.) up to which the visibility is to be the same as that on the axis, and by β_2 the extreme angle at which light, in the ideal case, will emerge from the lens. Let θ_1, φ_1 and θ_2, φ_2 be the corresponding values of θ and φ. Then only that light is useful which is emitted from the source between angles $-\theta_2$ and $+\theta_2$ with the horizontal. The intensity of the light emitted at the angle θ is proportional to $\cos \theta$, and, therefore (considering, say, the upper half of the lens) the light emitted between the angles o and θ bears to that emitted between o and θ_2 the ratio $\sin \theta / \sin \theta_2$. Now from Fig. 1 the ratio of the total amount of light emerging between angles o and β to that between angles

o and β_2 is to be $2\beta/(\beta_1+\beta_2)$, if $\beta<\beta_1$, and $1-\frac{(\beta_2-\beta)^2}{\beta_2^2-\beta_1^2}$ if $\beta>\beta_1$. The rays incident upon the first lens surface which divide the total energy in a given ratio must correspond with the emergent rays which have the same value for this ratio, neglecting all losses due to reflection and absorption. Thus, the conditions

$$\frac{\sin\varphi}{\sin\varphi_2}=\frac{\sin\theta}{\sin\theta_2}=\frac{2\beta}{\beta_1+\beta_2}, \text{ or } \frac{\sin\varphi}{\sin\varphi_1}=\frac{\sin\theta}{\sin\theta_1}=\frac{\beta}{\beta_1}, \text{ if } \beta<\beta_1,$$

and

$$\frac{\sin\varphi}{\sin\varphi_2}=\frac{\sin\theta}{\sin\theta_2}=1-\frac{(\beta_2-\beta)^2}{\beta_2^2-\beta_1^2}, \text{ or } \frac{\sin\varphi}{\sin\varphi_1}=\frac{\sin\theta}{\sin\theta_1}=\frac{\beta}{\beta_1}-\frac{(\beta-\beta_1)^2}{2\beta_1(\beta_2-\beta_1)}$$

if $\beta>\beta_1$

secure the distribution of light shown diagrammatically in Fig. 1 when the light source is small. The application of these formulæ enables the outline of the lens to be constructed geometrically in any given case. Applying them to the assumed conditions, with glass of refractive index 1·5, $a=100$ mm., aperture $=160$ mm., it is found that $\theta_2=38{\cdot}7$ deg., $\sin\theta_2=0{\cdot}6252$. Thus, dividing the light in the half lens into 12 equal parts, and sub-dividing the last section into four equal parts, and using the formula,

$$\tan\alpha=\frac{\mu\sin\varphi-\sin\beta}{\mu\cos\varphi-\cos\beta},$$

the following figures are obtained :—

β	Sin θ.	Sin φ.	θ	φ	μ Cos φ.	Tan α.	α.
Deg.			Deg.	Deg.			Deg.
0·000	0·0000	0·0000	0·000	0·000	1·5000	0·0000	0·000
0·972	0·0521	0·0347	2·987	1·988	1·4991	0·0699	4·000
1·944	0·1042	0·0695	5·981	3·983	1·4964	0·1412	8·039
2·917	0·1563	0·1042	8·992	5·981	1·4922	0·2140	12·078
3·889	0·2084	0·1389	12·028	7·983	1·4855	0·2882	16·078
4·861	0·2605	0·1737	15·099	10·005	1·4772	0·3656	20·084
5·861	0·3126	0·2084	18·215	12·028	1·4670	0·4454	24·010
6·947	0·3647	0·2431	21·388	14·071	1·4550	0·5271	27·795
8·149	0·4168	0·2779	24·631	16·135	1·4409	0·6098	31·375
9·514	0·4689	0·3126	27·960	18·215	1·4249	0·6920	34·684
11·132	0·5210	0·3473	31·397	20·324	1·4067	0·7706	37·618
13·241	0·5731	0·3821	34·964	22·463	1·3865	0·8350	39·862
13·924	0·5861	0·3907	35·880	23·000	1·3807	0·8425	40·113
14·733	0·5991	0·3994	36·807	23·544	1·3752	0·8449	40·193
15·787	0·6121	0·4081	37·745	24·088	1·3693	0·8364	39·910
18·333	0·6252	0·4168	38·695	24·631	1·3635	0·7499	36·867

It will be noted that α rises to a maximum and then falls, the change being fairly rapid at the edge of the lens; it is for this reason that the last section was subdivided into four parts.

To construct the outline of the lens, lines are drawn from the centre of the light source, making the angles θ with the horizontal; from the points in which these lines intersect the vertical corresponding to the inner lens surface lines are drawn, making the angles φ with the horizontal.

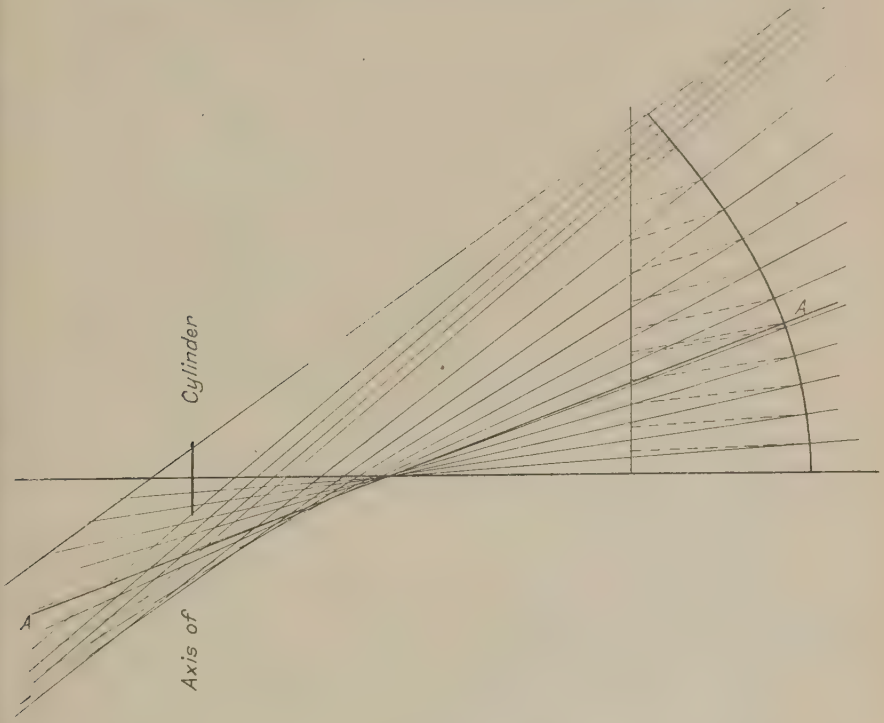

Fig. 3.

At a suitable point (determined by the minimum desirable marginal thickness for the lens) on the line inclined at φ_2, which represents the extreme ray, a straight line is drawn making an angle α_2 with the vertical; this will be a tangent to the surface. A point is taken on this tangent equidistant from the two refracted rays for which φ is 24·088 deg. and 24·631 deg., and a line drawn through it at the angle 39·910 deg. with the vertical: this will be the tangent to the surface where it meets the ray for

which φ is 24·088 deg. Similarly, from the point on this tangent equidistant from the rays φ=24·088 deg. and φ =23·544 deg., another tangent is drawn at the angle 40·193 deg. with the vertical, and so on. Finally a curve is drawn to touch these tangents. The nature of the surface is most clearly indicated by drawing the normals at the points for which the calculations have been made. This has been done in Fig. 3, which illustrates the lens for which the calculations are given

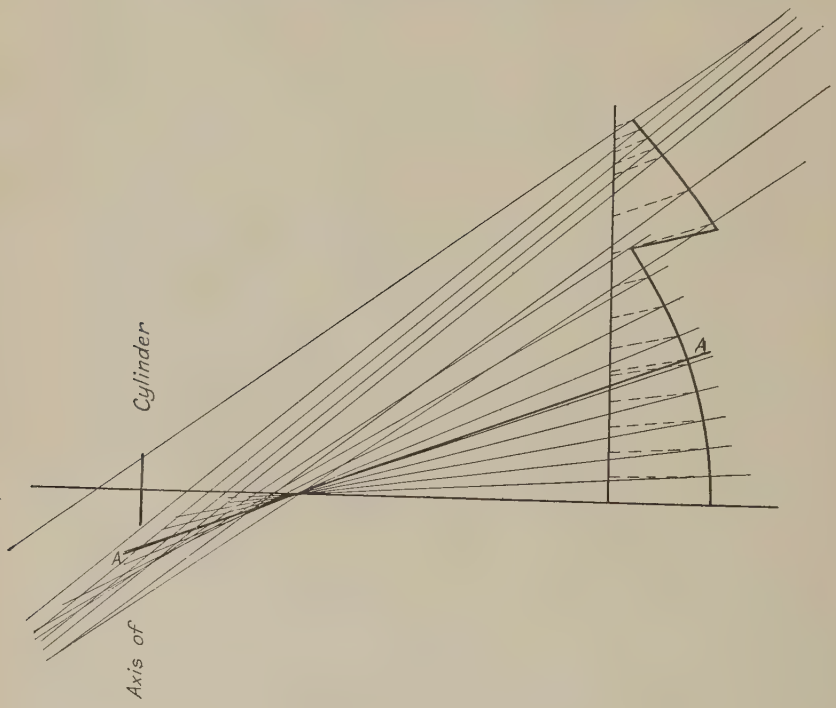

Fig. 4.

above. It will be noted that the part of the curve situated between the horizontal plane of symmetry and the line AA, which meets the surface where β=5 deg., and which therefore corresponds to uniform intensity in the refracted light, is of practically constant curvature and could be replaced by an arc of a circle without causing errors larger than are likely to be met with in manufacture. On going beyond this line, however, the curve rapidly becomes much flatter, being quite flat at some

position for which β lies between 13 deg. and 14 deg., and beyond that the curvature changes sign. Owing to the continual change in curvature such a lens would probably be made with greater accuracy if cast from a very carefully made mould than if ground.

Fig. 4 shows the same design so modified that the maximum thickness is rather less than half that shown in Fig. 3. The flat surface separating the central lens from the dioptric element is shown making such an angle with the horizontal that no light will be stopped, rays just entering the dioptric element from the top of the light source (the actual height of which is 16 mm.) being inclined at a greater angle with the horizontal, and rays from the bottom of the source which are refracted in to the air at the entrance edge of the central lens being inclined at a less angle with the horizontal than the lower edge of the dioptric element.

In consequence of the neglect of differences in the loss of light at different parts of the lens, due to reflection and absorption, there are likely to be slight divergencies of the light distribution from that shown in Fig. 1. These, however, will probably be very small compared with the errors present in the mould when the first lenses are cast, so that any corrections that may be required can be easily effected in the removal of the faults in the mould. In cases where the angles of deviation are appreciably larger than in the present example it may be necessary to apply a correction to the first calculations to compensate for the differences in the losses caused by reflection and absorption.

It is desirable to make sure that the distribution of the light will not be affected appreciably when the size of the light source is taken into account. The exact mathematical treatment of the general problem is troublesome, but any particular case is readily dealt with by calculating, for a small linear source placed vertically on the axis at a suitable distance above or below the plane of symmetry, the angles with the horizontal and vertical made by rays from this source before and after refraction. It is convenient to exhibit the results of such calculations by plotting sin θ against β as independent variable, where θ is the complement of the inclination of the ray to the light source before refraction, and β is the angle between the emergent ray and the plane of symmetry of the lens. The general character of the result obtained is indicated in Fig. 5, which gives the approximate distribution from a small linear

element 10 mm. from the flame centre—*i.e.*, 2 mm. outside the flame. Such a figure is most readily constructed by dividing up the useful energy produced by the element of the lamp which is being considered into a large number of equal parts, and finding the values of β which correspond to rays dividing these portions from one another. The erection on these segments of the β base of a number of equal rectangles gives an approximation to the required result. If the light source is symmetrical the corresponding element of the source on the opposite side of the median plane will give a distribution curve which is the reflection of the first, as shown by the lighter outline in Fig. 5. The distribution curve due to both of these elements,

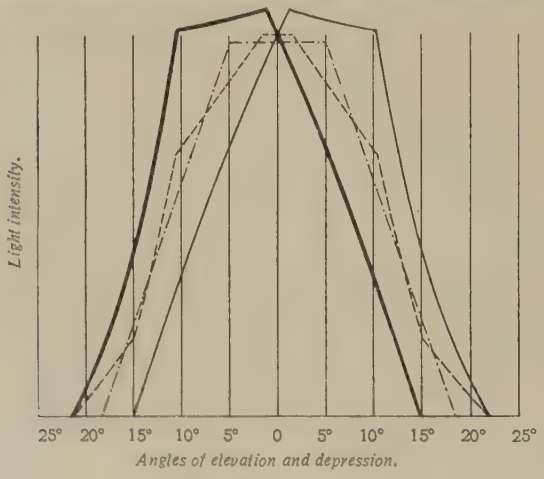

Fig. 5.

shown by a dotted curve, is obtained by taking a mean between the two previous curves. A comparison of this mean with the central distribution curve, which is chain dotted in the diagram, will indicate the direction and approximate amount of the corrections, if any, that are necessary.

ABSTRACT.

The principle on which lenses for securing a required distribution of light from a given source have been designed is illustrated by a two-dimensional example. The principle employed is to divide the incident and emergent energy into a number of equal parts, and compute the lens system so that the rays which separate off these portions of incident light from one another are refracted as rays which separate the corresponding portions of the emergent light.

The surfaces obtained are in general of varying curvature, and the lenses must, therefore, be moulded. It is shown how the effect of the finite size of the light source may be determined.

DISCUSSION.

Mr. R. S. Whipple asked how the results obtained with lamps designed on the lines discussed in the Paper compared with those previously employed, which had been evolved, he presumed, simply by the survival of the fittest? Were lenses of this type used as first cast or were the surfaces finally worked?

Mr. Smith stated in reply that the lamps previously in use gave very unsatisfactory results. Their performances varied greatly among themselves, and there did not appear to be any principle underlying their design. In many cases the lenses were not even symmetrically made, and the number of dioptric elements employed depended entirely on the whim of the maker. Some of these lenses had been ground, but he was not aware that they had been used except by the Admiralty. The new lenses, on the other hand, were giving very satisfactory results, despite the fact that some manufacturers had not yet quite realised the ideal conditions. The lenses were moulded and then, in some cases, glazed by what he believed was a secret process.

XXI. *The Choice of Glass for Cemented Objectives.* *By* T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received April 1, 1916.

It is well known that a telescope objective consisting of two component lenses can be so constructed as to be simultaneously free from chromatic and spherical aberrations and from coma, provided proper forms are given to the components. When these are to be cemented together the three conditions can only be satisfied if a careful choice is made of the kinds of glass used. Tables* have been published giving all the particulars necessary for the construction of such objectives when the corrections are required for an object at infinity. The present note deals with the changes in the kinds of glass to be employed when the objective is required to be free from these aberrations for objects at some finite distance.

The necessity for a change in the kinds of glass to be employed is readily demonstrated. The conditions† to be satisfied for freedom from spherical aberration and coma for magnification m (negative for an inverted image) are

$$A-m(3B+1)+m^2(3C+\varpi)-m^3B'=o,$$

and

$$B-m(3C+\varpi)+m^2(3B'+1)-m^3A'=o.$$

When the objective is thin, these conditions reduce to

$$B'-B=\frac{1+m}{1-m}(2+\varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$4C+2\varpi+1=\left(\frac{1+m}{1-m}\right)^2(5+2\varpi). \quad . \quad . \quad . \quad (2)$$

in consequence of the identical relations connecting the aberration coefficients. In the reduction the factor $1-m$ which occurs in both conditions has been rejected.

In an objective of the kind considered the only way in which the aberration coefficients can be varied, apart from a change in the kinds of glass used, to meet the different conditions when m is altered is by bending the lens as a whole. The changes in the coefficients consequent on such bending can be expressed in the form

$$B'-B=B'_0-B_0+2r(1+\varpi),$$

and

$$C=C_0+r^2(1+2\varpi),$$

* Gleichen: "Lehrbuch der Geometrischen Optik," p. 323.

† For notation employed *see* "Notes on the Calculation of Thin Objectives," "Proc." Ph. Soc., Vol. XXVII., p. 485.

where B_0, B'_0, C_0 and ϖ are constants depending only on the kinds of glass employed and r is a variable indicating the extent to which the objective is bent. The substitution of these expressions for $B'-B$ and C in equations (1) and (2) gives

$$B'_0-B_0+2r(1+\varpi)=\frac{1+m}{1-m}(2+\varpi), \quad . \quad . \quad . \quad (3)$$

and $$4C_0+2\varpi+1+4r^2(1+2\varpi)=\left(\frac{1+m}{1-m}\right)^2(5+2\varpi). \quad . \quad (4)$$

If, now, $(1+\varpi)^2 : (2+\varpi)^2=1+2\varpi : 5+2\varpi$ it would be possible to employ the same two kinds of glass to give a cemented doublet having the desired corrections for various magnifications. In fact, however,

$$(5+2\varpi)(1+\varpi)^2-(1+2\varpi)(2+\varpi)^2=1,$$

and thus the necessary condition can never be satisfied.

Equations (3) and (4) show that there are in general two magnifications for which there will be no spherical aberration and no coma with two given kinds of glass. The solutions may be written in the form :—

$$\frac{1+m}{1-m}=\pm(1+\varpi)\sqrt{(B'_0-B_0)^2(5+2\varpi)(1+2\varpi)+4C_0+2\varpi+1}$$
$$-(B'_0-B_0)(1+2\varpi)(2+\varpi),$$

and

$$2r=\pm(2+\varpi)\sqrt{(B'_0-B_0)^2(5+2\varpi)(1+2\varpi)+4C_0+2\varpi+1}$$
$$-(B'_0-B_0)(5+2\varpi)(1+\varpi).$$

These formulæ are readily applied to any system when B_0, B'_0, C_0 and ϖ are known. When these have not been calculated for a combination of two glasses the formulæ may with advantage be expressed differently. Using a notation already explained* $B'-B$ and C may be put into the forms

$$B'-B=\frac{x-y}{1+2\varpi}+\frac{1+\varpi}{1+2\varpi}\{2r(1+2\varpi)+3y-(K-K')(3+2\varpi)\}, \quad (5)$$

and $$C=KK'(R+R'+1)^2-KK'(1+2\varpi)+(KR-K'R')^2$$
$$+r\{3y-(K-K')(3+2\varpi)\}+r^2(1+2\varpi)$$
$$=-\frac{1+2\varpi}{4}+KK'(R+R'+1)^2+\frac{x^2}{4}-\frac{5+2\varpi}{4(1+2\varpi)}(x-y)^2$$
$$+\frac{1}{4(1+2\varpi)}\{2r(1+2\varpi)+3y-(K-K')(3+2\varpi)\}^2, \quad (6)$$

* *See* "Thin Objectives," p. 489, equation (3).

where $$y=K(1+R)-K'(1+R')$$

and $$x=K-K'+y\varpi.$$

It at once follows that

$$(B'_0-B_0)(1+2\varpi)=x-y,$$

and that

$$(B'_0-B_0)^2(5+2\varpi)(1+2\varpi)+4C_0+2\varpi+1=x^2+4KK'(R+R'+1)^2.$$

The magnifications for which the corrections can be made are, therefore, given by

$$\frac{1+m}{1-m}=\pm(1+\varpi)\sqrt{x^2+4KK'(R+R'+1)^2}+(2+\varpi)(y-x), \quad (7)$$

and the corresponding curvatures of the cemented surface are

$$y+\tfrac{1}{2}(2+\varpi)\{\pm\sqrt{x^2+4KK'(R+R'+1)^2}-x\}. \quad . \quad . \quad (8)$$

Formulæ (7) and (8) are in a convenient form for numerical use. To determine suitable combinations of glasses for various distances two refractive indices may be selected and the calculations performed for varying values of K and K'. When a set of values corresponding with the available range of optical glasses has been thus determined one of the refractive indices may be altered and a fresh set of values obtained. The results of series of these calculations in which one of the refractive indices is constant throughout are conveniently plotted in two dimensions, with the other refractive index and the ratio of the powers of the component lenses as the variables for Cartesian co-ordinates. Contours for which m has a constant value can be drawn in the chart by interpolation among the values obtained from the calculations for the two dimensional series of points in exactly the same way as contours are drawn from field observations in surveying. Instead, however, of marking off lengths in one direction in proportion to $-K/K'$, it is more convenient to make them proportional to $\log(-K/K')$. By so doing it becomes possible, when a series of charts such as has just been described has been constructed, to find in a few minutes whether any combination of two glasses out of the many kinds which may be available will enable the required conditions to be satisfied. For it is only necessary to

prepare on translucent paper a chart of these glasses taking as ordinates and abscissæ their values of μ and $\log \nu$, — the μ referring to the wave-length for which correction is required, and the ν being calculated for the wave-lengths for which the focal lengths are to be equal—on the same scale as that used in plotting refractive indices and $\log(-K/K')$ in the contour maps. The glass chart is then placed over one of these maps with the corresponding lines μ=constant superposed. Let each available glass of the refractive index for which the particular map was constructed be brought in turn over the point for which $\log(-K/K')=0$. If in any case the contour for the desired magnification passes through the point representing another glass, the glass thus obtained when combined with that for which the two diagrams are adjusted will give a cemented objective having the desired correction. The justification for thus using the two diagrams after the manner of a slide rule lies in the relation $-K/K'=\nu/\ '$ whence $\log(-K/K')=\log\nu-\log\ '$.

In constructing these contour maps much time may be spent fruitlessly in making calculations for points which only give imaginary values for m. This may be avoided by first of all plotting the curve which separates the regions where the roots are imaginary from those where they are real. From equation (7) this boundary is given by

$$x^2+4KK'(R+R'+1)^2=0\ ;$$

i.e., the boundary consists of the two curves

$$x=2(R+R'+1)\sqrt{-KK'} \quad . \quad . \quad . \quad . \quad (8)$$

and

$$x=-2(R+R'+1)\sqrt{-KK'} \quad . \quad . \quad . \quad (9)$$

Suppose that μ, K and K' are given. These equations then serve to determine μ'. The solution presents no difficulties, as x involves μ' only to the second order and R' involves it to the first order. The equations are, therefore, quadratic in μ', and the boundary is traced out by solving these for a series of values of K and K'. Fig. 1 gives a general view of the part of the boundary in the neighbourhood of the origin when one refractive index is 1·5 and Fig. 2 is drawn for refractive index 1·6. Each of these diagrams in reality contains the results of calculations for two separate cases, those parts above the line

$\log(-K/K')=0$, relating to systems in which the lens of fixed index is the negative component, and those below this line to

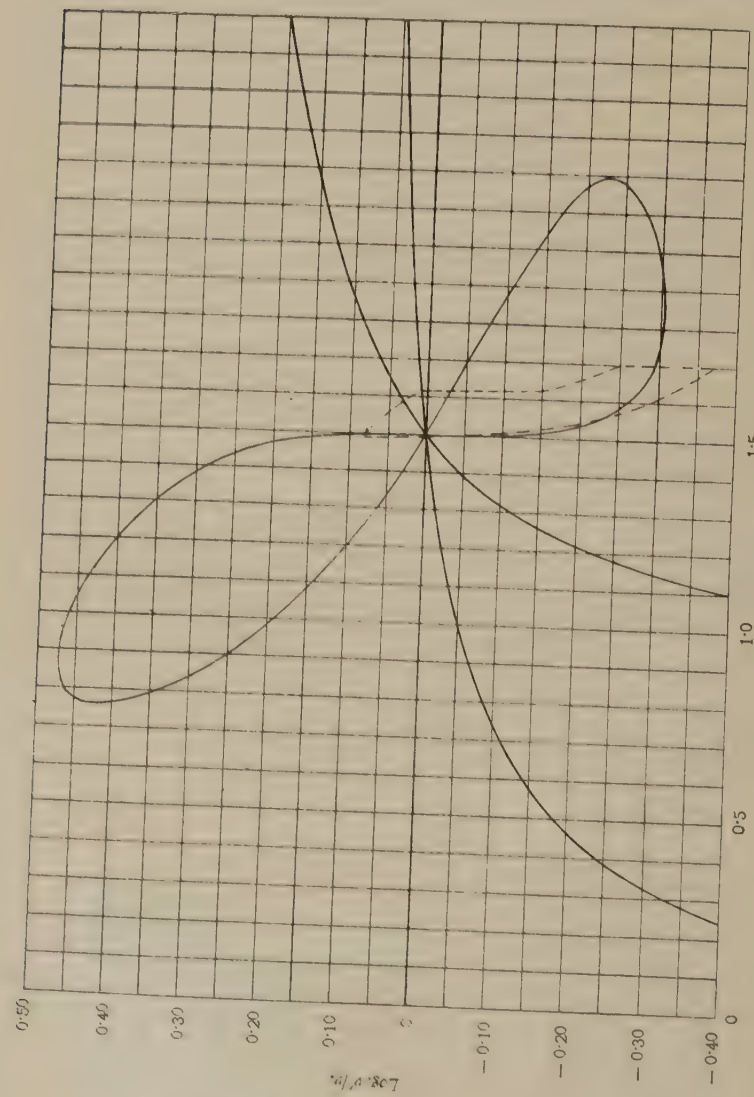

FIG. 1.

systems in which it is the positive component. The two are conveniently included together in the one diagram, as the

boundaries in the one case are the natural continuations of those in the other.

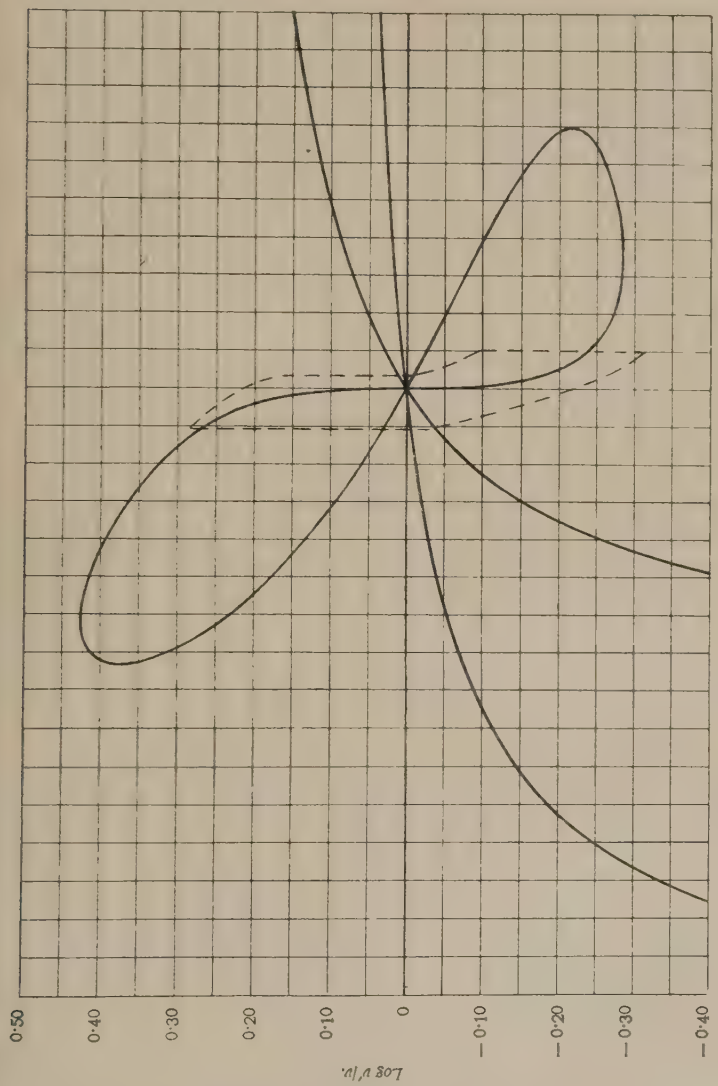

FIG. 2.

The curve given by equation (8) is closed and resembles a written figure 8 in shape. The branch in the upper part of the

diagram is longer than that in the lower, and both loops decrease in size as the fixed refractive index is increased. The curve of equation (9) consists of two branches which are asymptotic to $\mu=0$ and pass through the point representing the fixed glass. In the upper part of the map there are two asymptotes perpendicular to the lines μ = constant. The positions of these asymptotes for various values of the refractive index of the fixed glass are as follows :—

Fixed index.	log $(-K/K')$	
1·4	0·11362	0·37359
1·5	0·10694	0·32453
1·6	0·10102	0·28728
1·7	0·09575	0·25791
1·8	0·09102	0·23407

There are other branches to the boundary, but they are of no practical interest.

To distinguish between the real and imaginary regions it is sufficient to take a single case. Let the two refractive indices be equal, so that, so far as spherical aberration and coma are concerned, the two glasses may be regarded as one. The magnifications for which there is correction are then μ and $1/\mu$, the lens taking the well-known meniscus form in which the object or image is at the centre of curvature of one surface, which coincides with the aplanatic point of the other surface. Thus the line $\mu=\mu'$ is one of the contours which are to be traced, and the regions in which this line lies are those for which real solutions exist.

It is not very convenient to plot the contours for equal differences in m, as the range corresponding to real images, or negative values of m, goes from 0 to $-\infty$. The contours may with advantage be plotted for equal differences in $\frac{1+m}{1-m}$ Equation (7) shows that this quantity is always finite for finite curvatures of the lens surfaces, and it has the further advantage that the substitution for m of its reciprocal merely alters the sign of $\frac{1+m}{1-m}$, so that for a given magnification one contour gives the case of a lens with the crown component leading, and the corresponding contour of opposite sign gives a lens with the flint component leading. The range of values through which this quantity goes for all real images is from -1 through

zero to +1. The following table will be found useful in making use of the charts :—

TABLE 1.

Value of $\frac{1+m}{1-m}$	Value of m.	Value of $\frac{1+m}{1-m}$	Value of m.
0·00	−1	0·55	−0·29032
0·05	−0·90416	0·60	−0·25000
0·10	−0·81818	0·65	−0·21212
0·15	−0·73913	0·70	−0·17647
0·20	−0·66667	0·75	−0·14286
0·25	−0·60000	0·80	−0·11111
0·30	−0·53846	0·85	−0·08108
0·35	−0·48148	0·90	−0·05263
0·40	−0·42857	0·95	−0·02564
0·45	−0·37931	1·00	0
0·50	−0·33333		

The dotted outlines in Figs. 1 and 2 indicate approximately the area that can be covered on the contour maps by the kinds of optical glass that have so far been manufactured when the glass chart is given as great a movement parallel to the lines μ = constant as the range of glass allows. Obviously the regions of most interest are those between the infinite branches and the figure of eight projections through which the line $\mu = \mu'$ runs. The principal features of the contours in these parts can be ascertained with some probability of accuracy from simple general reasoning.

For this purpose it is convenient to consider an analogous case in which the conditions can be readily visualised. Suppose, then, that the conditions are expressed in terms of the more familiar case of surveying. Horizontal distances in two directions at right angles—say, North and South, East and West—take the place of μ and $(-K/K')$ respectively. Heights will be the analogue of values of $\frac{1+m}{1-m}$. The case is not, however, quite that of an ordinary survey, for there are two vertical distances involved. Imagine, then, that the charts relate to a cave, the one height being that of the roof, the other that of the floor. The regions where the roots are imaginary correspond with solid rock, where there is neither roof nor floor. The curve separating the real and imaginary regions gives the horizontal extension of the cave, the walls of which at points on the curve are vertical. In this particular case we also know that, along the North and South line through the extremity of the recess, the height of the roof and the depth of the floor from

the datum level are constant and equal. Thus if a number of East and West vertical sections are projected in a North and South direction and drawn superimposed on one another they will intersect in two points, the heights of which will be—taking the case of refractive index 1·5 for instance—plus and minus five. The contours with which we are most concerned are plus and minus one and those between these. Clearly there must be points on these contours between the line $\mu=1\cdot5$ and the walls on either hand. As the sections are taken nearer to the closed end of the cave the wall which becomes tangential to the plane of constant height will become flatter, the level at which it is perpendicular tending to the contour of zero height.

It would be expected that with the falling width of the cave and the increased steepness of the walls the maximum height of the cave would increase, and that at the extremity the opening would become a vertical line of infinite length. Nothing can be said as to the level at which the walls of the infinite branch are vertical.

It is evident that all the contours which are of more practical importance pass through the line extremity of the cave and emerge through the corresponding opening opposite, and that they tend to crowd together as the cave becomes more confined in width. In those parts where the walls are vertical somewhere between the contours plus and minus one the contours for all negative magnifications will tend to be close together, so that a change in the kinds of glass may not be necessary in practice.

It is reasonable to suppose that the level at which the wall running nearly North and South is vertical will gradually change from zero height, which it should have near the extreme end, to appreciably different levels in the more open parts. The various contours will thus touch the curve on the plan marking the extreme horizontal extent of the cave. On one side of these points of contact the contours will lie below the perpendicular part of the wall, and on the other side they will be above it. The shape of the extreme boundary evidently serves as an approximate indicator of the shape of the contours close to the wall, but nothing can be said without detailed calculations of the directions the contours will take when the projecting bluff of rock has been passed.

Reference to the diagrams shows the general features to which this reasoning leads. Fig. 3 shows for $\mu=1\cdot5$ a number of vertical sections of that part of the cave which lies to the

right of the plane $\mu=\mu'$ by planes which have the values of $\log \nu'/\nu$ given in the diagram. The contours for which $\frac{1+m}{1-m}=+1$, 0 and -1 derived from these sections are given in Fig. 4. The curve on the right, which is hardly distinguishable from the contour $\frac{1+m}{1-m}=-1$ is the boundary between real and imaginary regions. The figures on the right of this boundary give the contours which touch the boundary at the positions indicated. The contours for real images appear very close together, so much so that it is natural to inquire whether the same kinds of glass will not in practice give, for any negative magnification, an objective indistinguishable from one made of the theoretically necessary glasses notwithstanding the mathematical impossibility. To decide this point a special case may be considered. With a given pair of glasses let the magnifications for which correction is theoretically perfect be m' and m''. Let an objective be made from these which gives no coma at magnification m, the lens itself acting as stop. The condition for this is

$$B'-B=\frac{1+m}{1-m}(2+\varpi).$$

The amount of spherical aberration is then measured by

$$\tfrac{1}{4}\{4C+2\varpi+1-\left(\frac{1+m}{1-m}\right)^2(5+2\varpi)\}.$$

This may be shown to be equal to

$$-\frac{1}{4(1+\varpi)^2}\left(\frac{1+m}{1-m}-\frac{1+m'}{1-m'}\right)\left(\frac{1+m}{1-m}-\frac{1+m''}{1-m''}\right).$$

From the diagram a case may be selected at random : a possible combination is

$$\frac{1+m'}{1-m'}=1, \quad \frac{1+m''}{1-m''}=-3.$$

If a lens were made from these glasses in the way suggested for $m=-1$, the spherical aberration would amount to about 0·26, a value which cannot be neglected for many purposes. It may be concluded that although the same two glasses can be satisfactorily used for a fair range of magnifications, it will for some purposes be necessary to secure glasses which lie on the

theoretical curve well within the limits of the extreme curves for real images.

When the two conditions cannot be satisfied, and it is decided to satisfy one which will give two roots and keep the amount of the second aberration as small as possible, it is desirable if the second glass is taken from the real region to keep as close as possible to the boundary which separates the real from the imaginary regions, as otherwise the roots for

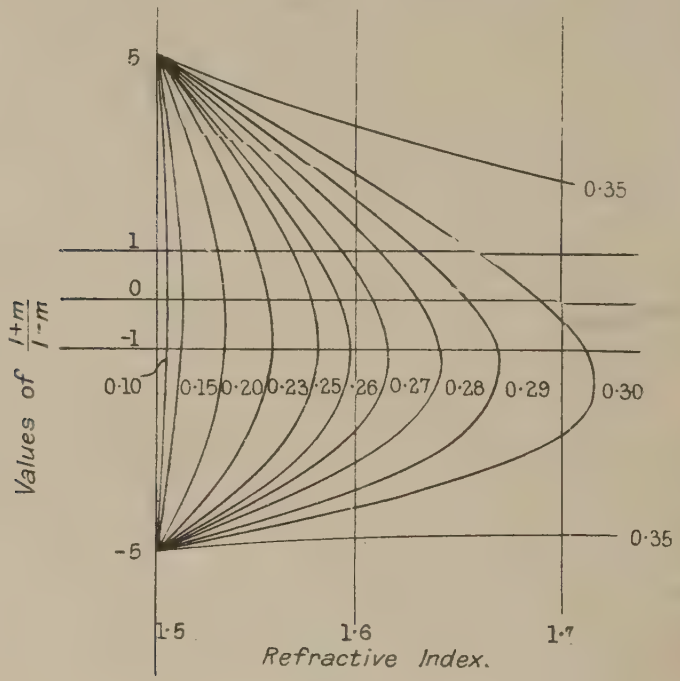

Fig. 3.

the selected glasses may prove to be imaginary. This is readily seen on considering the case of spherical aberration. When both glasses have the same refractive index it is impossible to secure freedom from spherical aberration for any negative magnification. There must, therefore, be a region in the cave in which the spherical aberration condition leads to imaginary roots for real images. The boundary to this region is in fact close to the boundary of the imaginary region with which we have been dealing, and the contours that are traced on the

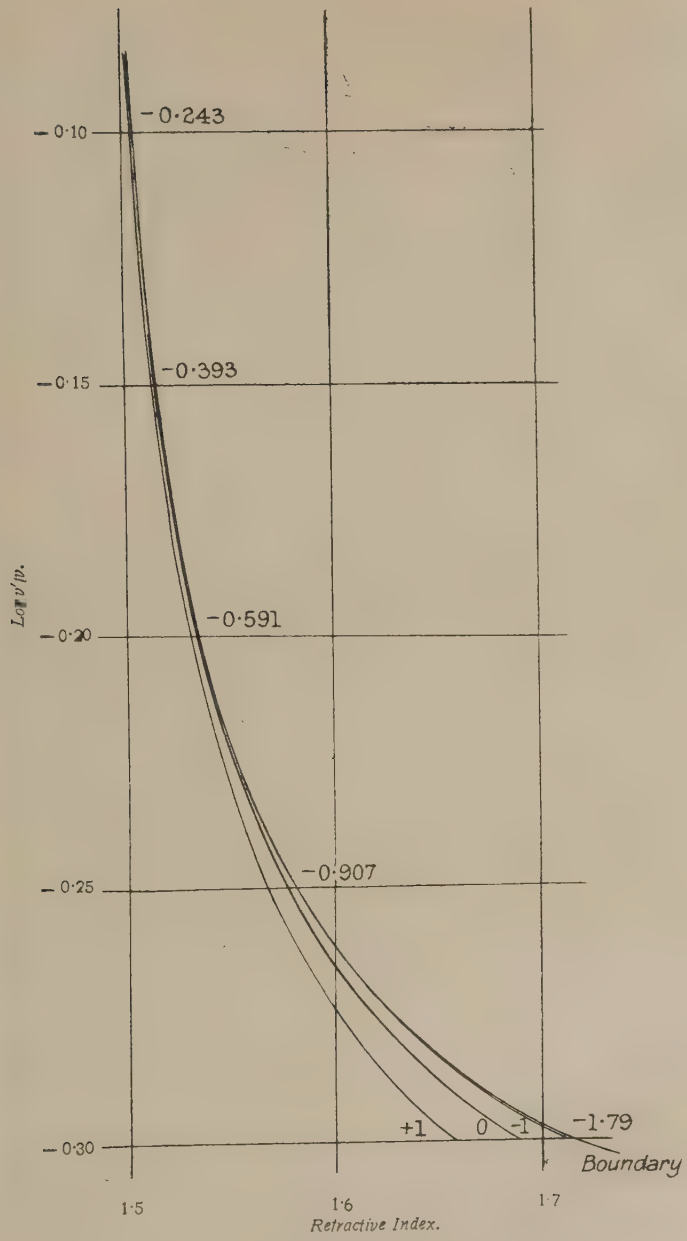
−0·243
−0·393
−0·591
−0·907
−1·79
+1
0
−1
Boundary
−0·10
−0·15
−0·20
−0·25
−0·30
Log v/w.
1·5
1·6
1·7
Retractive Index.

FIG. 4.

charts necessarily lie in the space between these two imaginary regions.

When the kinds of glass necessary to secure the corrections with sufficient accuracy are not available, a triple objective becomes necessary. The author has shown* that the solution in this case also depends upon quadratic equations, and thus there is the possibility of finding imaginary roots with such lenses as well as with doublets. A case in which the solution fails is obtained by taking equal refractive indices for the two glasses. The formulæ already obtained are readily applied to this further case. From equations (6) of the previous investigation it is clear that $\mathbf{B}'_0-\mathbf{B}_0$ is a linear, and $\mathbf{C}_0$ a quadratic function of θ. Consider the functions f_1 and f_2 where

$$f_1=\mathbf{B}'_0-\mathbf{B}_0-(B'_0-B_0)(1-2\theta)$$

and

$$f_2=(4\mathbf{C}_0+2\varpi+1-A')-(4C_0+2\varpi+1-A')(1-2\theta)^2.$$

When $\theta=0$, the relations $\mathbf{B}'_0-\mathbf{B}_0=B'_0-B_0$ and $\mathbf{C}_0=C_0$ must hold. When $\theta=1$, the necessary relations are $\mathbf{B}'_0-\mathbf{B}_0=-(B'_0-B_0)$ and $\mathbf{C}_0=C_0$. When $\theta=\frac{1}{2}$ from symmetry $\mathbf{C}$ is a minimum, *i.e.*, $\mathbf{C}=\mathbf{C}_0$, and $\mathbf{B}_0=\mathbf{B}'_0$. In such a symmetrical system it is not difficult to prove that $4\mathbf{C}+2\varpi+1=A'$ where the A' relates to one-half of the system as actually combined with a similar lens to produce the compound system. It follows that $f_1=f_2=0$ when $\theta=0$, $\frac{1}{2}$ and 1 and hence f_1 and f_2 are always identically zero.

The equation giving m in terms of θ is evidently

$$\frac{1+m}{1-m}=\pm(1+\varpi)\sqrt{(1-2\theta)^2\{(B_0'-B_0)^2(5+2\varpi)(1+2\varpi)+4C_0+2\varpi+1-A'\}+A}$$
$$-(B_0'-B_0)(1-2\theta)(1+2\varpi)(2+\varpi).$$

Rearranging this as an equation for θ in terms of m it is found that θ is real if

$$\left(\frac{1+m}{1-m}\right)^2\{(B_0'-B_0)^2(1+2\varpi)(5+2\varpi)+4C_0+2\varpi+1-A'\}$$
$$-A'\{(B'_0-B_0)^2(1+2\varpi)+(4C_0+2\varpi+1-A')(1+\varpi)^2\}$$

* "Thin Objectives," p. 495.

is positive. Over the range of suitable glasses for objectives the spherical aberration is positive for parallel incident light when the first surface is plane; that is to say, A' is positive, and since the region within which

$$(B_0'-B_0)^2(1+2\varpi)(5+2\varpi)+4C_0+2\varpi+1$$

is negative is that in which triple objectives are likely to prove most useful, the triple form will first break down when $\frac{1+m}{1-m}=1$. When the two refractive indices are equal $B_0=B'_0$ and $4C_0+2\varpi+1=A'$, so that the above quantity is zero. When the difference between the refractive indices is small, it has a positive value if the refractive index of the less dispersive medium is less than that of the more strongly dispersive medium, and is negative if the reverse is the case. Thus one region in which the triple solution breaks down is within the cave, and bounded by the line $\mu=\mu'$, the region lying on that side of this line which is not practically important. It is not difficult to verify that on the other side of this line no imaginary region is reached until the area which can be covered by useful optical glasses is passed. Thus the triple form of objective can be employed in any practical case in which the available kinds of glass do not enable a satisfactory cemented doublet to be made.

It may be noted that the contours giving the kinds of glass for which the two conditions are satisfied can be used for other purposes. For example, in a doublet lens they mark off the region where the one aberration is positive from that in which it is negative when the second aberration is removed. In triple lenses they separate the region where the two lenses of one kind of glass are of the same sign from that in which they are of different signs for one form of solution. Other uses to which they can be put readily suggest themselves in various problems which are encountered in the course of designing optical instruments.

TABLE II.

$\log \nu/\nu'$	K.	K'.
0·05	9·1955	−8·1955
0·10	4·8621	−3·8621
0·15	3·4240	−2·4240
0·20	2·7097	−1·7097
0·25	2·2849	−1·2849
0·30	2·0048	−1·0048
0·35	1·8073	−0·8073
0·40	1·6614	−0·6614

The author desires in conclusion to express his indebtedness to Miss Dale, Mr. Trump and Mr. Bell for their assistance in the calculations from which the contours have been plotted.

ABSTRACT.

The strict fulfilment of the mathematical conditions for freedom from colour, spherical aberration and coma, for objects at varying distances from a thin cemented doublet lens, necessarily demands a change in the kinds of glass as the position of the object is changed The Paper describes a method by which the proper glasses can be determined by using a glass chart on translucent paper, in conjunction with diagrams calculated for the purpose, as a slide-rule. With most glasses the solution to the problem is imaginary, and formulæ are given for calculating the boundary separating real and imaginary solutions by solving a quadratic equation. In practice it is found that the curves for all real images lie very close to one another. Notwithstanding this, a numerical example shows that there will be a perceptible difference in the definition obtained if in place of the theoretically necessary type a kind of glass is substituted which does not differ greatly from the other, and lies on a curve giving perfect correction for another real magnification. It is further shown that when no real solution is forthcoming for a double lens a real solution will in practice be obtained by adopting a triple objective made of the same glasses.

XXII.—*The Correction of Chromatic Aberrations when the External Media are Dispersive. By* T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received April 1st, 1916.

Summary.

When one of the external media of a lens system is dispersive it is not possible to ensure the absence of differences in the size and position of images of all objects formed by light of different wave-lengths. The degree to which correction can be carried is investigated, and formulæ are given by which the power and position of the external surfaces of a system can be found when the type of correction to be adopted is given.

1. It will be convenient to denote the four quantities which determine the behaviour of the paraxial rays in the optical system by the letters A, B, C, D. The power of the system will be denoted by A; B and C determine the position of the extreme lens surfaces relatively to the cardinal points of the system; for a thin system D is zero. The standard relation connecting these four quantities will be taken as $BC-AD=1$.

If the refractive indices of the first and last media be μ and μ', the distances from the first and last refracting surfaces of the conjugate points for which the magnification is N, will be x and x', where

$$x=-\frac{\mu}{A}\left(B-\frac{1}{N}\right),$$

and

$$x'=\frac{\mu'}{A}(C-N),$$

both distances being measured from the surfaces in the direction from the first to the last surface.

The conditions for complete colour correction (*i.e.*, no colour differences in the size and position of the image of any object) are that B, C, and the ratios $\mu:\mu':A$ should be independent of the colour of the light.

2. If only one external medium is dispersive these conditions cannot all be satisfied. Under these circumstances it may be shown that a thin system will not form an image of any object of finite size free from colour. Putting the conditions down generally we find the change of magnification given by

$$\frac{\delta N}{N}=(1-N)\left\{\frac{\delta\mu}{\mu}-\frac{\delta A}{A}\right\}. \quad . \quad . \quad . \quad . \quad (1)$$

The separation of the images is equal to

$$\frac{\mu'(1-N)}{A}-\frac{(\mu'+\delta\mu')(1-N-\delta N)}{A+\delta A}$$

$$=\frac{\mu'(1-N)}{A}\left\{-\frac{\delta\mu'}{\mu'}+\frac{\delta A}{A}+\frac{\delta N}{1-N}\right\}$$

$$=\frac{\mu'(1-N)}{A}\left\{\frac{\delta\mu}{\mu}-\frac{\delta\mu'}{\mu'}-\frac{\delta N}{N}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by (1). Thus the position and magnification with a thin system cannot be the same for two colours unless the external refractive indices are proportional for the two colours, proving the result stated above.

3. Equation (2) shows that the inconsistency of the two-colour conditions when one medium is dispersive and the other is not, is of such magnitude as to render a thin system quite unsuitable for all practical purposes.

4. The next simplest arrangement to a thin system is two thin systems separated by air. In this case D is independent of colour. The solution to which the conditions for both corrections for magnification N lead is

$$\frac{\delta A}{A}=\frac{C}{C-N}\frac{\delta\mu}{\mu}+\frac{NB}{NB-1}\frac{\delta\mu'}{\mu'} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\delta B=\frac{NB-1}{C-N}\frac{\delta\mu}{\mu}+B\frac{\delta\mu'}{\mu'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\delta C=C\frac{\delta\mu}{\mu}+\frac{C-N}{NB-1}\frac{\delta\mu'}{\mu'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

obtained by putting $\delta x=\delta N=0$ in the equations of the first paragraph. These results may be verified by noting that they satisfy

$$B\delta C+C\delta B-(BC-1)\frac{\delta A}{A}=0$$

showing that $\delta D=0$, and by substituting the values of the differentials in the formulæ for the differences in size and position of the images of an object placed so that the magnification for one of the two colours considered is N'. The results so obtained are

$$\frac{\delta N'}{N'}=(N'-N)\left\{\frac{1}{C-N}\frac{\delta\mu}{\mu}+\frac{B}{NB-1}\frac{\delta\mu'}{\mu'}\right\}$$

and $$\delta x' = \frac{\mu'(N'-N)}{A}\left\{\frac{C-N'}{C-N}\frac{\delta\mu}{\mu} - \frac{N'B-1}{NB-1}\frac{\delta\mu'}{\mu'}\right\}$$

$$= \frac{\mu'(N'-N)}{A}\left\{\frac{\delta\mu}{\mu} - \frac{\delta\mu'}{\mu'} - \frac{\delta N'}{N'}\right\}$$

a result which corresponds exactly with (2) when $N=1$.

5. The more general case of correction for magnification N when D is not restricted to be independent of the colour may be investigated as follows :–

Write the fundamental equations in the form,

$$x = -\frac{\mu D}{AD}\left(B - \frac{1}{N}\right)$$

and $$x' = \frac{\mu' D}{AD}(C-N).$$

Then $$B = \frac{1}{N} - x\frac{AD}{\mu D}$$

and $$C = N + x'\frac{AD}{\mu D},$$

so that $$AD = BC - 1 = \left(\frac{1}{N} - x\frac{AD}{\mu D}\right)\left(N + x\frac{AD}{\mu' D}\right) - 1,$$

or since in general both A and D are finite

$$AD + \frac{N}{x'}\mu' D - \frac{1}{Nx}\mu D + \frac{\mu D \cdot \mu' D}{xx'} = 0.$$

Differentiate this applying the conditions

$$\delta x = \delta x' = \delta N = 0.$$

Then $$\delta(AD) = \left(\frac{1}{N} - \frac{\mu' D}{x'}\right)\frac{\delta(\mu D)}{x} - \left(N + \frac{\mu D}{x}\right)\frac{\delta(\mu' D)}{x'}$$

or $$\frac{\delta(AD)}{AD} = \frac{C}{C-N}\frac{\delta(\mu D)}{\mu D} + \frac{NB}{NB-1}\frac{\delta(\mu' D)}{\mu' D} \quad . \quad . \quad (6)$$

Now differentiate the expressions for B and C applying the same conditions as before, and substituting from (6) for $\delta(AD)$. The results

$$\delta B = \frac{NB-1}{C-N}\frac{\delta(\mu D)}{\mu D} + B\frac{\delta(\mu' D)}{\mu' D} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $$\delta C = C\frac{\delta(\mu D)}{\mu D} + \frac{C-N}{NB-1}\frac{\delta(\mu' D)}{\mu' D} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

immediately follow.

It will be observed that equations (6), (7) and (8) are derived from (3), (4) and (5) by writing AD, μD, and $\mu' D$ in all cases for A, μ and μ' respectively. In this more general case we shall thus have

$$\frac{\delta N'}{N'} = (N'-N)\left\{\frac{1}{C-N}\frac{\delta(\mu D)}{\mu D} + \frac{B}{NB-1}\frac{\delta(\mu' D)}{\mu' D}\right\}$$

and $$\delta x' = \frac{\mu' D(N'-N)}{AD}\left\{\frac{C-N'}{C-N}\frac{\delta(\mu D)}{\mu D} - \frac{N'B-1}{NB-1}\frac{\delta(\mu' D)}{\mu' D}\right\}$$

$$= \frac{\mu'(N'-N)}{A}\left\{\frac{\delta\mu}{\mu} - \frac{\delta\mu'}{\mu'} - \frac{\delta N'}{N'}\right\} \text{as before.}$$

6. The result involving the quantity $\frac{\delta\mu}{\mu} - \frac{\delta\mu'}{\mu'} - \frac{\delta N'}{N'}$, which thus holds in all cases, might have been written down at once from the consideration that the ratio of the distance between two image points on the axis to the distance between their conjugates is μ'/μ times the product of the transverse magnifications for the two points. Taking one of these points as that for which the two chromatic aberrations are corrected, the preceding result at once follows. The unsuitability of thin lens systems follows from the fact that the position for which the aberrations are corrected is that occupied by the lens itself.

7. An interesting case is obtained when $N=0$. Equations (6) and (8) then reduce to A/μ=constant and $C\mu'/\mu$=constant. In other words, the focal length in the dispersive medium in which the rays are parallel is to be constant, and the rays of different colours incident parallel to the axis at any given point of the first refracting surface are to meet the last surface at distances from the axis proportional to the ratio of the refractive index of the first to that of the final medium for their respective colours. The chromatic conditions will thus be satisfied by choosing the position of the last surface to suit one condition and adjusting its power to satisfy the other.

8. As another example the case of the system considered in 4, with air as the final medium, may be taken. Equation (5) gives C/μ=constant, showing that the type of correction required by the first system is independent of the magnification for which the whole is to be used. It would therefore be

possible in this case to substitute in the same position different thin lenses for the second system, and obtain correction for various positions of the object. If t be the axial separation of the two groups the conditions are satisfied by making the power of the first system equal to $(1-\mu p)/t$, and of the second to $\frac{1}{t}+\frac{1}{x'}-\frac{1}{\mu p}\left(\frac{1}{t}+\frac{N}{x'}\right)$, where p has the same value for different colours. The power for the complete system is $(\mu p-N)/x'$.

The power of the first system may be written in the form $\frac{1-p}{t}+\frac{(1-\mu)p}{t}$ showing that if this lens would be itself rendered achromatic by replacing the dispersive external medium by air, and if the ratio of the radius of curvature of the first surface to the focal length of the lens in air be $(1-p)/p$, the lens will be properly corrected for this purpose, whatever the nature of the first medium with which it is actually used. In particular, such a lens could be used in conjunction with suitable second systems for the formation of the images of objects in dispersive liquids. Since p is not subject to any limitations it is evident that any achromatic lens may be so used, and its value of p used to determine a suitable second system. A number of other interesting solutions can be readily obtained. Any practical case can be quickly worked out with the help of equations (6), (7) and (8).

9. When all the chromatic conditions cannot be satisfied it is not necessarily the case that the best corrections will be obtained by applying the conditions that the image should be fully corrected for some one magnification. Where a considerable range of magnifications is to be employed it may be preferable to have the images of various colours formed on the principal ray for the standard colour. The position of the stop will evidently enter into this condition. Let the stop be placed in such a position that its magnification for the complete lens is d. Then if the distances from the first and last surfaces of the stop and of its image are S and S', the relations

$$S=-\frac{\mu}{A}\left(B-\frac{1}{d}\right) \text{ and } S'=\frac{\mu'}{A}(C-d)$$

will be satisfied.

The conditions to be applied are evidently

$$\delta x=0 \text{ and } \frac{\delta x'}{x'-S'}=\frac{\delta N}{N}.$$

These lead to

$$-d\left\{\left(\frac{\delta\mu}{\mu}-\frac{\delta A}{A}\right)(NB-1)+N\delta B\right\}$$
$$=(C-N)\left(\frac{\delta\mu'}{\mu'}-\frac{\delta A}{A}\right)+\delta C=d\frac{\delta N}{N}. \quad (9)$$

10. Equation (9) will be satisfied for all values of N if

$$d\left(\frac{\delta\mu}{\mu}-\frac{\delta A}{A}\right)=C\left(\frac{\delta\mu'}{\mu'}-\frac{\delta A}{A}\right)+\delta C,$$

and

$$\frac{1}{d}\left(\frac{\delta\mu'}{\mu'}-\frac{\delta A}{A}\right)=B\left(\frac{\delta\mu}{\mu}-\frac{\delta A}{A}\right)+\delta B,$$

which reduce to

$$S'\delta B=\delta(\mu' D)$$

and

$$-S\delta C=\delta(\mu D).$$

The differences in the size and position of the images of various colours are given by

$$\frac{\delta N}{N}=\frac{NB-1}{dB-1}\frac{\delta(\mu D)}{\mu D}-\frac{C-N}{C-d}\frac{\delta(\mu' D)}{\mu' D}=\frac{\delta x'}{x'-S''},$$

and object points are represented in the image plane for the standard colour by circles of diameter $\frac{\delta N}{N}$ times the diameter of the stop in the image space, with their centres at the image point for the standard colour.

11. This effect in the image plane for the standard colour may be compared with the corresponding results given by correcting according to equations (6), (7) and (8). In this case the diameter of the diffusion circles when the magnification for the standard colour is N', is

$$\frac{N'-N}{N'-d}\left\{\frac{C-N'}{C-N}\frac{\delta(\mu D)}{\mu D}-\frac{BN'-1}{BN-1}\frac{\delta(\mu' D)}{\mu' D}\right\},$$

or

$$\frac{A\delta x'}{\mu'(N'-d)}$$

times the diameter of the stop in the final medium, and the magnification in this plane determined by the centres of these circles is

$$1+\frac{N'-N}{N'-d}\left\{\frac{C-d}{C-N}\frac{\delta(\mu D)}{\mu D}-\frac{dB-1}{NB-1}\frac{\delta(\mu' D)}{\mu' D}\right\}$$

times that for the standard colour.

12. When equation (9) is only satisfied for the one magnification N the extra degree of freedom present may be taken into account by regarding $\frac{\delta N}{N}$ as an independent variable.

The equations for the changes in the focal length and position of the foci for different colours are then

$$\frac{\delta(AD)}{AD}=\frac{C}{C-N}\frac{\delta(\mu D)}{\mu D}+\frac{NB}{NB-1}\frac{\delta(\mu' D)}{\mu' D}+\frac{C-dNB}{(C-N)(NB-1)}\frac{\delta N}{N},$$

$$\delta B=\frac{NB-1}{C-N}\frac{\delta(\mu D)}{\mu D}+B\frac{\delta(\mu' D)}{\mu' D}-\frac{dB-1}{C-N}\frac{\delta N}{N},$$

$$\delta C=C\frac{\delta(\mu D)}{\mu D}+\frac{C-N}{NB-1}\frac{\delta(\mu' D)}{\mu' D}+\frac{C-d}{NB-1}\frac{\delta N}{N}.$$

When the magnification is N' for the standard colour the diffusion circles in this image plane will be of diameter

$$\frac{N'-N}{N'-d}\left[\frac{C-N'}{C-N}\left\{\frac{\delta(\mu D)}{\mu D}-\frac{dB-1}{NB-1}\frac{\delta N}{N}\right\}-\frac{N'B-1}{NB-1}\left\{\frac{\delta(\mu' D)}{\mu' D}+\frac{C-d}{C-N}\frac{\delta N}{N}\right\}\right]+\frac{(C-N')(BN'-1)(N-d)}{(C-N)(BN-1)(N'-d)}\frac{\delta N}{N}$$

times the diameter of the stop in the image space, and the magnification for the centres of these circles will be

$$1+\frac{N'-N}{N'-d}\left\{\frac{C-d}{C-N}\frac{\delta(\mu D)}{\mu D}-\frac{dB-1}{NB-1}\frac{\delta(\mu' D)}{\mu' D}-\frac{(C-d)(dB-1)}{(C-N)(NB-1)}\frac{\delta N}{N}\right\}$$

times that for the standard colour. The results of the preceding sections can be found from these formulæ by giving the appropriate values to $\frac{\delta N}{N}$.

The best way of correcting can be determined from the last two expressions on taking into consideration the range of magnifications for which the system is to be used, and the relative magnitudes of the field of view and the angular aperture.

XXIII. *Note on the Use of the Auto-Collimating Telescope in the Measurement of Angles.* By J. GUILD, *A.R.C.S., D.I.C., F.R.A.S., from the National Physical Laboratory.*

RECEIVED APRIL 1, 1916.

THE purpose of this note is to draw attention to certain sources of error which exist in some of the methods commonly adopted for the optical determination of angles, and which have not, as far as the author is aware, been pointed out before.

As an example of this type of measurement consider the case in which it is desired to compare an angle under test (say, the right angle of a totally reflecting prism) with a standard angle of the same nominal value of which the error is known. There

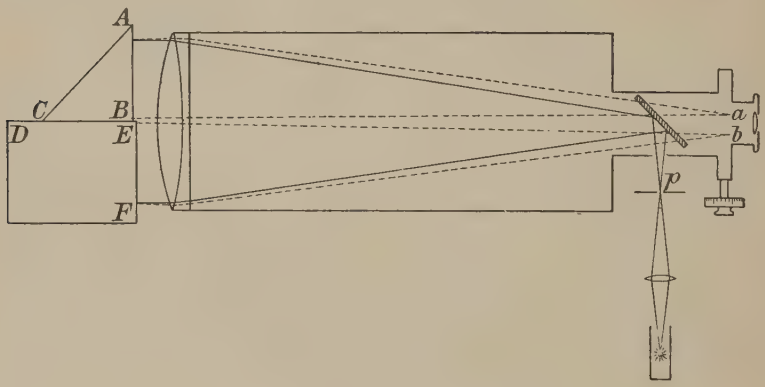

FIG. 1.

are several methods by which this may be accomplished, but the one which at first sight seems most attractive is the following :—

One face, B C, Fig. 1, of the prism is placed against the face D E of the block containing the standard angle DEF, and the surfaces are worked together until the interference fringes formed by the air film are as broad as possible and parallel to the edge B C. The inclination of A B to E F is then determined by means of some form of auto-collimator. The instrument which is employed at the Laboratory for measurements of this nature consists of an astronomical telescope of 3 in. aperture and 125 cm. focal length. A piece of unsilvered plane parallel glass, *m*, situated 5 or 6 cm. in front of the focal plane,

serves to reflect light from a small illuminated pinhole, p, along the axis of the telescope.

The latter is first focussed on infinity and the position of p is then adjusted so that an optical flat placed in front of the objective and at right angles to the axis of the telescope returns the light to form an image of p in the focal plane of a Zeiss micrometer eyepiece. Reverting to Fig. 1 it is clear that if the planes A B and E F are not exactly parallel, two images a and b, will be formed, one from each plane. By gently tapping the prism with a pencil it can be rotated on the block until the line joining a and b is parallel to the traverse of the micrometer (initially adjusted to be approximately parallel to the edges A B and E F). The distance $a\ b$ is then measured and converted into angle. From this and the known value of $\hat{E}$ the angle B can at once be deduced.

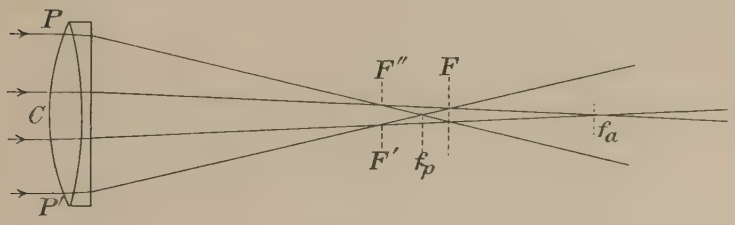

FIG. 2.

It will be seen that the images a and b are formed by light from opposite halves of the objective, and that the line of division of the latter is perpendicular to the direction of the displacement to be measured. The objection to this arrangement is twofold. In the first place, if the objective is not wholly free from spherical aberration, a spurious displacement in opposite directions is given to each image. Fig. 2 represents the case of a lens undercorrected for spherical aberration. If parallel light is incident on the whole lens the peripheral rays are brought to a focus at f_p, while rays near the axis converge to f_a. The circle of least confusion which represents the mean focus is situated at F, say, somewhere between f_p and f_a. If now light from the upper half, P C, of the lens be cut off, as is the case for the image from E F (Fig. 1), the circle of least confusion is no longer at F but at F′. Similarly, if C P′ be obscured as is the case for the image from A B (Fig. 1), the focus is at F″. Hence, even if the return beams leave A B and E F accurately parallel,

the images a and b will be separated by an amount $F'F''$, which is only zero if the residual spherical errors are quite negligible.

In the second place, even with a perfectly corrected objective, the method depends for its success on the accurate focussing of the eyepiece on the images. In Fig. 3 let a and b be the true positions of the images from the two surfaces. If the focal plane of the eyepiece is a little behind or in advance of the true position, a and b will be seen as circles of confusion with centres situated at a' and b' or a'' and b'', as the case may be. In an actual case where the focal length of the object glass is large the sharpness of the images is sensibly uniform over an appreciable range. It is difficult to give a true indication of the magnitude of effects of this nature in a diagram in which dimensions are necessarily exaggerated, but the author has frequently found it to be possible, by racking the eyepiece out

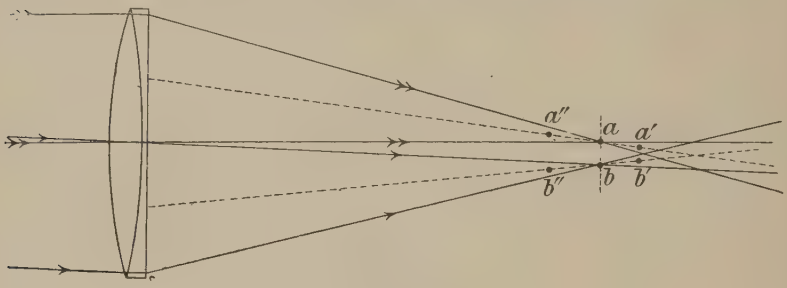

FIG. 3.

and in, to cause the images to move over a considerable range to and from each other without the definition being sensibly spoilt. It is, therefore, impossible to say that the correct displacement between a and b has been obtained.

The unsatisfactory nature of all exact focussing methods was pointed out by Prof. Abbe, who showed that in many cases it could be surmounted by using "Telecentric" adjustment—an arrangement in which all parts of the image or images in the focal plane are formed by narrow pencils whose central rays are parallel to the axis. Unfortunately, telecentric adjustment is not possible in the type of measurement under discussion.

It is clear that the two sources of error mentioned above depend on the fact that the plane of division of the objective is perpendicular to the direction of the displacement which it is desired to measure, and the objections apply equally to any other method in which the objective is divided in this manner.

There are, however, certain cases in which matters can be arranged so as to avoid this difficulty. There are two methods open to us in the case already considered. If an optical flat is available, the standard block and the prism under test are placed side by side on the flat (Fig. 4), care being taken in each case to have the interference fringes running absolutely normal to the edges B B′ and E E′. The auto-collimator is then presented normally to the face A B′ and F E′, and the block (or prism) is gently tapped until the two images are in the same vertical plane. Their vertical displacement is then measured. In this case the plane of division of the objective is parallel to the plane of the measured displacement and neither spherical aberration nor depth of focus materially affects the displacement in this plane. Displacements due to these causes at

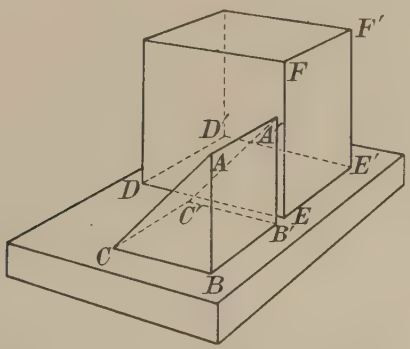

Fig. 4.

right angles to the plane of measurement are, of course, immaterial, and are nullified in the preliminary adjustments. With this method, or any other involving the assumption of zero angle between two faces (such as those of the prism and block in Fig. 1, or each of those with the optical flat in Fig. 4), it is necessary to use considerable care in the adjustment of the fringes. A difference in the thickness of the air film corresponding to one interference fringe in a length of about 5 cm. is equivalent to 1 sec. of arc. The fact that there is parallax between the fringes and the film as the eye is moved about and the further difficulty of accurately interpreting their appearance when viewed through refracting surfaces, as has often to be done, renders it somewhat difficult to be sure that no error comparable with the degree of accuracy aimed at

(1 sec. approximately in most of this work) remains from this adjustment.

A method which does not involve the adjustment of interference fringes is to mount the two angles which are to be compared, one above the other, as in Fig. 5. In the figure the prism is shown sitting on the side of the block, but it is, of course, essential to set it on a small levelling table which rests on the block in order that the edge B B′ may be adjusted to be parallel to E E′. This may be done by any of the ordinary methods. The collimator is then placed normal to the faces B A′ and E F′, so that the smaller one, B A′, is symmetrically placed from left to right when viewed through the telescope with eyepiece removed. The unsymmetrical portion of the face E F′ to the right of the dotted line is obscured by a suitable screen. When the adjustments have been properly made the

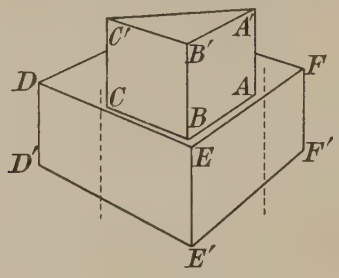

Fig. 5.

images from B A′ and E F′ should be in a plane perpendicular to E E′. By measuring their displacement in this plane the small angle between the faces B A′ and E F′ is obtained. The base carrying the blocks is now cautiously rotated, so as to bring the images from C B′ and D E′ into the field of the collimator, the portion of D E′ to the left of the dotted line being screened off. As an alternative to rotating the base, the collimator may be moved round. The inclination of the faces C B′ and D E′ can now be obtained. From these two differences the value of the unknown right angle can be deduced. Although more laborious than the method of Fig. 4, inasmuch as the adjustment of the levelling screws is somewhat tedious and two measurements have to be made instead of one, this arrangement has been found to be preferable when the highest accuracy is desired.

S. D. Chalmers and H. S. Ryland (" Transactions," Optical

Society, 1904-1905, p. 34) describe a number of attractive methods for determining the angles of prisms without standard comparison blocks. In two cases, however, the arrangements present the objectionable feature which has already been discussed. These are the cases of 90 deg. and 120 deg. In the first of these a parallel beam of light is incident approximately normally on the hypotenuse face (Fig. 6, *a*). The rays such as A, B, C, D, E incident to the left of the 90 deg. edge leave the prism in a direction D E, while the rays incident to the right of the edge come out in the direction D′ E′. The inclination of D E to D′ E′ depends only on the value of the right angle, and is measured by the auto-collimator. It will be seen that the two images in the field of the eyepiece are produced by

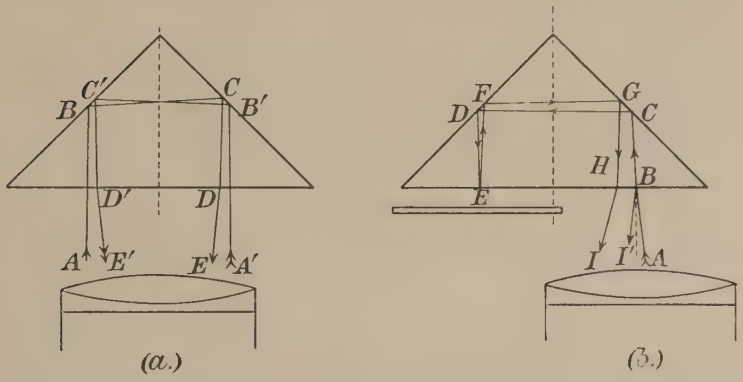

FIG. 6.

light coming entirely from the right and left halves of the objective respectively, so that the uncertainties due to depth of focus and also to any spherical aberration which may exist are present. The method can readily be modified to avoid this disadvantage. One half of the prism is screened off (Fig. 6, *b*), and the auto-collimator is placed opposite the other half. When the necessary adjustments have been made the two images which will be seen in the field are produced by rays which have traversed the paths A, B, I′ and A, B, C, D, E, F, G, H, I respectively. The inclination of H I to B I′ depends, as in the previous case, on the value of the right angle only. In this case, however, corresponding rays in each image pass through the same portion of the object glass, and the small displacement of the

images in the field of the eyepiece is unaffected by focussing or by spherical aberration.*

All the reflections except those at B and E are total, so that the images are of nearly equal intensity and quite bright enough for accurate measurements. The case of 120 deg. can be modified in the same way, but it is necessary in this case to silver the two faces containing the obtuse angle since the reflections at these faces are not total, and the image which undergoes multiple reflection would be too faint to observe.

This note may be concluded with a general observation on the relative merits of the auto-collimating methods and the more direct methods of measuring prism angles. The former are undoubtedly susceptible of greater accuracy where every possible precaution is taken to eliminate all sources of error such as those herein discussed. Nevertheless with a good quality circle reading to, say, 5 secs. by two or more micrometer microscopes, it is possible to determine prism angles correct to about 3 secs. or even less with reasonable care. While this accuracy may be exceeded if desired by auto-collimating methods, the necessary trouble which has to be taken in order to do so renders the process somewhat tedious.

With regard to a choice of method for determining angles in any given case it should be borne in mind that the maximum accuracy which it is useful to secure depends on the purpose for which it is intended to use the prism and the quality of the glass. For instance, in the case of a prism designed for measurements of refractive index, in which the deviation of narrow pencils passing through different portions of the material differ by as much as 10 secs., as is sometimes the case in prisms whose total performance is quite satisfactory, a knowledge of the angles to an accuracy greater than about 5 secs. is clearly superfluous, and the time spent in obtaining it is wasted.

* It should be noted that this arrangement can only be used with convenience when the pyramidal error of the prism is small, *i.e.*, when the normal to the hypotenuse face is in or near the plane containing the normals to the other faces. It is clear that if this is not so the ray returning from E is deflected either downwards or upwards, while BI′ is deflected in the opposite direction. Hence the line of separation of the two images will not be in the plane containing the normals to the faces bounding the right angle. The component of the separation parallel to this plane will not, however, be affected, and it is this that should be measured. But it is difficult to obtain satisfactory measurements with most types of micrometer eyepiece if the line joining the two images is not nearly parallel to the traverse of the micrometer. The method is, therefore, only suitable for use in cases where this condition is fulfilled. Fortunately, among prisms of good quality the pyramidal errors encountered are rarely sufficient to cause trouble.

ABSTRACT.

The measurement of angles by means of the autocollimator resolves itself into the measurement of the distance between two images produced in the focal plane of a micrometer eyepiece. In most cases the light forming these images passes through portions of the object glass on opposite sides of a diameter. It is shown that, when this diameter is perpendicular to the direction of the displacement to be measured, uncertainty and error are introduced on account of any residual spherical aberration of the object glass and the depth of focus of the telescope. One or two particular cases are discussed in which it is shown how this may be obviated.

DISCUSSION.

Dr. R. S. Willows asked why the Abbé type of autocollimating eyepiece, such as is used on Pulfrich refractometers, &c., was not employed in the arrangement described in the Paper.

The Author explained that there was a difficulty in adapting this method of illumination to a micrometer eyepiece.

XXIV. *The Viscosity of Colloidal Solutions.* By Emil Hatschek.

Received March 30, 1916.

In his recent Guthrie Lecture on "Some Problems of Living Matter" ("Proc." XXVIII., Part II., p. 99), Mr. Hardy, in discussing gelation, makes the following remarks: "According to the theory of Einstein and Hatschek a decrease in fluidity is due to an increase in the volume of the colloid particles following upon an intake of water from the continuous medium. I do not think so simple a theory covers the facts. Their formula fails except in the case of the simplest hydrosols and suspensions."

This statement is based on, and may help to perpetuate, two misconceptions, which it seems desirable to clear up. In the first instance, no formula for the viscosity of a sol can by any possibility be applicable to a gel or even to the transition stage of gel formation. This is perfectly obvious, without any assumptions whatever on the structure of gels—a subject which is still highly debatable—if the remarkable elasticity of gels, to which Mr. Hardy refers, is remembered. As this elasticity is absent in the sol state, there must be a profound difference between the two systems, which at once precludes the possibility of representing their mechanical properties by the same formula.

In the second instance, Einstein's and my formula was never intended to apply to such systems as Mr. Hardy exclusively discusses, systems in which a transfer of water from the continuous into the disperse phase is possible. Although the methods used by Einstein * and by myself† in deducing the formula are entirely different, the assumptions regarding the constitution of the suspension are identical—viz., that the disperse phase consists of *rigid* spheres of such sizes as to conform to Stokes' formula—that these spheres in the aggregate occupy only a small portion of the total volume, and that they do not influence one another.

This definition—which is difficult to realise even in the case of true suspensions or suspensoids—can certainly not be held to apply at all to the class of colloids which form the contents of cells. The universal view is that these systems consist of

* A. Einstein, "Ann. d. Physik," XIX., 289, 1906.
† E. Hatschek, "Koll. Zeitschr." VII., 301, 1910.

two *liquid* phases, and the generic title "Emulsoids," which is very generally accepted for them, expresses this view. The disperse phase in such systems—*e.g.*, albumin, gelatin, silicic acid or rubber sols—is supposed to consist of deformable aggregate formed by the dispersed substance in association with a large and varying proportion of the dispersion medium.

For such sols, or generally for systems consisting of two liquid phases, I have given a formula which was published very soon after my suspensoid formula.* It applies to systems in which the disperse phase occupies the greater part of the total volume, so that its particles must assume a polyhedral shape. If such a system is sheared, deformation of the polyhedra must occur, the only force tending to restore them to their original shape being the interfacial tension between the two phases, which is necessarily small in any stable system. It is, therefore, probable that, at rates of shear exceeding a certain minimum, the structure remains deformed, so that the whole shear takes place in the thin films of continuous phase separating the polyhedra of disperse phase. The viscosity of the whole system can then be expressed by the following equation :

$$\eta' = \eta \frac{\sqrt[3]{A}}{\sqrt[3]{A} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the symbols mean :—

A = the ratio total volume/volume of disperse phase.

η' = viscosity coefficient of system.

η = viscosity coefficient of continuous phase (dispersion medium).

It will be noticed that the value of η' increases very rapidly as A approaches unity—*i.e.*, as the disperse phase occupies more and more of the total volume ; and this fact, as well as the shape of the curve defined by the equation, agrees perfectly with experimental data.

It will also be noticed that neither the *viscosity of the disperse phase* nor the *degree of dispersity* (the absolute size of the particles) appear in the equation. The absence of these factors, however, while surprising at first sight, is what renders the formula directly applicable to emulsoids, since both of them are unknown in these systems.

As regards A, the ratio total volume/volume of disperse phase, and experimental verification of the formula, which presumes that this value is known, a distinction must be made

* E. Hatschek, "Koll. Zeitschr.," VIII., 34, 1911.

between emulsions proper—*i.e.*, dispersions of one liquid in another in which it is practically insoluble (*e.g.*, petroleum in dilute soap solution) and emulsoids sols.

In the former the ratio A is known from the volumes used in preparation, or can easily be determined afterwards by breaking up the emulsion. The formula has been tested for such an emulsion, the viscosity being determined in a modified Couette apparatus at a low rate of shear.*. It will be seen that, by a simple transformation of the formula, the ratio A can be calculated from one viscosity determination. If, as is usually done, we make η the viscosity coefficient of the medium=unity we obtain :—

$$A=\left(\frac{\eta'}{\eta'-1}\right)^3, \quad \ldots \ldots \ldots \quad (2)$$

from which A, and, therefore, the volume percentage of disperse phase ($100/A$) can be directly calculated. In the case of the petroleum-soap water emulsion the ratio found by direct determination was 79·8 per cent. of oil, while that calculated from the viscosity by applying equation (2) was 76·7 per cent., an agreement which must be considered quite satisfactory in view of the destruction which such emulsions undergo even at low rates of shear.

As regards the emulsoids, say, of sols of proteins, gum arabic or rubber, the case is somewhat different, since we cannot determine the volume of disperse phase directly. We can, however, again calculate it from a viscosity determination, and find that each gramme of dissolved substance, *at one particular concentration*, produces n cubic cm. of disperse phase. This single result naturally does not constitute any check on the formula, and to obtain one we must make some assumption as to the amount of dispersion medium associated with the dispersed substance at different concentrations. The simplest and most natural assumption is that the association factor is constant at least over a certain range. If this is true, we must obtain *a constant when dividing the volumes of disperse phase, as calculated from the viscosities of sols of various concentrations, by the amounts dissolved to obtain these concentrations.*

Such calculations have been carried out by myself † on

* E. Hatschek, "Colloids and their Viscosity," "Trans." Faraday Soc., LVII., 1913.

† *Loc. cit.*, IV., p. 56.

published viscosity measurements of other observers, and by Miss H. Chick on viscosity measurements on the proteins of serum carried out by herself.* A very satisfactory constancy over a wide range was found in all cases. Thus, the relative viscosity of casein sol (water at the same temperature=1) rises from 6·12 at 6·05 per cent. to 23·72 at 9·39 per cent., while the association factors for these two extremes are 9·638 and 9·358. Miss Chick has also shown that the relative values of association for different proteins agree perfectly with their behaviour under various conditions involving an alteration in the amount of associated water—*e.g.*, in salting out—at higher temperatures, &c.

It is possible to obtain evidence of the correctness of the formula, or at least of the views on which it is based, in another manner. Some colloids, like rubber, which can be dissolved in a number of different solvents, show very different viscosities in them at equal concentrations. If the views here discussed are correct this should mean that rubber is associated with a large volume of the dispersion medium when showing high viscosity and with a small volume in the sols exhibiting low viscosity. Now, it is known that rubber swells considerably, but to a very different extent, in the liquids in which it ultimately dissolves. It is only reasonable to assume that the amounts of associated solvent calculated from the viscosity, and the amounts absorbed in the preliminary swelling, should show some agreement. An investigation in this sense has been carried out by Kirchhof.† He determined the viscosity of rubber sols in different solvents, and calculated from them the amount of associated solvent according to formula (2). A comparison of the figures so obtained with the figures for solvent taken up in preliminary swelling—published by another observer, Posnjak—shows "remarkable agreement," not only in their relative order, but even in absolute amount.

To sum up—

1. No conceivable viscosity formula can cover the stage of gelation.

2. The suspensoid formula deduced by Einstein and independently by myself applies only (and imperfectly) to systems in which the disperse phase consists of rigid spherical particles.

* H. Chick, "Biochem. Journal," VIII., 261, 1914.
† F. Kirchhof, "Koll. Zeitschr.," XV., 30, 1914.

3. A formula for emulsoids—*i.e.*, systems in which the disperse phase is also liquid—published in 1911, has since received a considerable amount of experimental confirmation.

ABSTRACT.

The author, in reply to some remarks made by Mr. W. B. Hardy in the course of his Guthrie lecture, points out (*a*) that no viscosity formula can cover the stage of gel formation, since the change from a liquid with only slight anomalies to a system having many properties of an elastic solid necessarily precludes this, and (*b*) that the formula given by Einstein, and independently by himself, for the viscosity of a suspension of rigid spherical particles, does not in any event apply to systems such as discussed by Mr. Hardy, which belong to the class known as emulsoids. In these both phases are quite generally assumed to be liquid, and an expression for the viscosity of such systems was deduced by the author in 1911, from a consideration of the deformation which a homogeneous polyhedral structure must undergo when sheared. The expression contains only the viscosity of the system and the phase ratio, but neither the interfacial tension of the phases nor the viscosity of the disperse phase. It can be tested directly for emulsions of two immiscible liquids; in the case of emulsoids, where only the weight of dispersed substance is known, some assumption has to be made about the relation between the volume of disperse phase formed—*i.e.*, the degree of hydration or solvation—and the weight dissolved. The simplest assumption, that this ratio is constant, has been tested by the author, and by Dr. Harriette Chick, for a number of sols, and is found to hold good over a considerable range. The formula has also been tested by F. Kirchhof in a different way, by comparing the degree of solvation—*i.e.*, the amount of solvent taken up by the disperse phase in different solvents—with the amount of the solvent taken up by the substance (indiarubber) in the swelling which precedes dispersion. Remarkable agreement both in the order and in the absolute amounts in different solvents was observed.

DISCUSSION.

Dr. A. Griffiths said he was in emphatic agreement with the conclusion at the end of paragraph two. He mentioned the case of a gelatine which would not run through a capillary tube, but which could be made to do so when broken up by stirring with a stick. He thought experiments on rigidity as well as viscosity should be made.

The Author, in reply to a remark by the President, stated that accidental double refraction was sometimes observed in the liquid phase.

XXV. *A Recalculation of Some Work on Diffusion by Dr. A. Griffiths and Others.* By A. GRIFFITHS, *D.Sc.*

RECEIVED APRIL 24, 1916.

THE Papers which require certain corrections are :—

I. "Determination of the Coefficient of Diffusion of Potassium Chloride by an Analytical Method." By A. Griffiths, J. M. Dickson and C. H. Griffiths, "Proc." Phys. Soc., Vol. XXVIII., Part I., Dec. 15, 1915.

II. "On the Coefficient of Diffusion in Dilute Solutions." By B. W. Clack. "Proc." Phys. Soc., Vol. XXVII., Part I., Dec. 15, 1914.

III. "On the Temperature Coefficient of Diffusion." By B. W. Clack. "Proc." Phys. Soc., Vol. XXIV., Part I., Dec. 15, 1911.

IV. "On the Coefficient of Diffusion." By B. W. Clack. "Proc." Phys. Soc., Vol. XXI. May, 1909.

In the experiments described on the above Papers diffusion takes place along a vertical tube, or a series of vertical tubes.

The lower ends of the tubes are kept in contact with a solution at a constant concentration, and the upper ends are generally kept in contact with water. It is shown that the liquid in the tubes is in movement, and reference is made to the velocity of the liquid through which diffusion takes place ; but what is meant by the velocity of the liquid is unfortunately not explained ; and it would have been better to deal with the *velocity of the water component of the liquid*, to which a clear meaning can be attached.

The majority of the symbols given on page 73 of the first Paper on the list retain their specifications, but the meaning of V should be taken to represent the velocity of the water-component of the solution, and not the velocity of the liquid.

With the new meaning of V, equations (1) and (2) on page 73 still hold good ; but equation (4) on page 74 is incorrect ; for c represents the total mass of potassium chloride proceeding from the reservoir per unit area per second, and its place is taken by water and by water only.

Thus $V \times$ mass of water in one continuous current of solution at end of tube $= c\delta$.

$$\therefore\ V(D-N) = c\delta$$

and

$$v = c\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (5) is now redundant.

Equation (3), p. 74, should have been.

$$\frac{N-n_0}{c}\left\{1-\frac{v(N+n_0)}{2c}\right\}=\frac{L}{K}.$$

Finally, equation (6), p. 74, takes the form

$$K=\frac{Lc}{(N-n_0)}\cdot\frac{1}{1-\frac{1}{2}(N+n_0)\delta}.$$

It may be mentioned incidentally that when the tops of the tubes are in contact with water the corrected equation (4) may be obtained by noting that the volume-movement of the water downwards at the upper ends of the tubes must be equal to the diminution in volume of the solution in the reservoir due to the escape of the potassium chloride.

The value of K given in the first Paper must be multiplied by

$$\frac{1-\frac{1}{2}(D-N)(N-n_0)\delta}{1-\frac{1}{2}(N+n_0)\delta}.$$

The correcting value varies from 1·005 to 1·01, and has a mean value of 1·007. Thus the corrected value of K_{18} in the case of a solution of potassium chloride containing 0·2237 gramme to the cubic centimetre (a 3N solution) is 1·696 ; and the corrected value of $K_{18\cdot5}$ is 1·715.

In the last Paper on the list at the head of this note, equations (1), (2) and (3) are unaffected ; but equation (4) becomes

$$K=\frac{Li}{N\left(1-\frac{N\delta}{2}\right)(1-\delta)}.$$

Thus the values for K in all Mr. Clack's Papers must generally be multiplied by

$$\frac{\left\{1-\frac{N\delta}{2}(D-N)\right\}\ \{1-\delta(D-N)\}}{\left(1-\frac{N\delta}{2}\right)(1-\delta)}.$$

Mr. Clack will correct the whole of his summarised results himself ; but I have his permission to deal with the case of a 2·7 normal solution of potassium chloride. In the case of feeble solutions the multiplier approaches unity, but in the case under consideration its value is 1·062.

Thus his corrected value of K_{18} is $1{\cdot}628\times10^{-5}$; and the corrected value given by Paper I. is only 4 per cent. greater than that obtained by Mr. Clack.

I may be allowed to remark that the value of K_{18} deduced from my Paper on Diffusive Convection, in the "Proc." Phys. Soc., Vol. XXVIII., Part I., December, 1915 (to which fortunately no correction is necessary), is $1{\cdot}617\times10^{-5}$. This is only 0·7 per cent. less than Mr. Clack's corrected value.

There is no doubt that, as thus recalculated, the results given by the three methods agree within the limits of experimental error; and my opinion is that the true value of the coefficient of diffusion lies close to those given by Mr. Clack's gravitational method, and by my convective method.

PUBLICATIONS OF THE PHYSICAL SOCIETY.

THE SCIENTIFIC PAPERS
OF THE LATE
SIR CHARLES WHEATSTONE, F.R.S.

Demy 8vo, cloth. Price 8s.; to Fellows, 4s.

Uniform with the above.

THE SCIENTIFIC PAPERS
OF
JAMES PRESCOTT JOULE, D.C.L., F.R.S.

Vol. I. 4 Plates and Portrait, price 12s.; to Fellows, 6s.
Vol. II. 3 Plates, price 8s.; to Fellows, 4s.

PHYSICAL MEMOIRS.

PART I.—VON HELMHOLTZ, On the Chemical Relations of Electrical Currents. Pp. 110. *Price 4s.; to Fellows, 2s.*

PART II.—HITTORF, On the Conduction of Electricity in Gases; PULUJ, On Radiant Electrode Matter. Pp. 222. *Price 8s.; to Fellows 4s.*

PART III.—VAN DER WAALS, On the Continuity of the Liquid and Gaseous States of Matter. Pp. 164. *Price 8s.; to Fellows, 4s.*

REPORT ON RADIATION AND THE QUANTUM-THEORY
By J. H. JEANS, M.A., F.R.S.

Price 6s.; to Fellows, 3s. Bound in Cloth, 8s. 6d.; to Fellows, 5s. 6d.

PROCEEDINGS.

The "Proceedings" of the Physical Society can be obtained at the following prices:—

Vol. I. (3 parts) bound cloth, 15s.
Vols. II., IV., V., XXIII., XXV., XXVI. & XXVII. (5 parts each), bound cloth, 23s.
Vols. III., VI. to XII. & XXII. (4 parts each), bound cloth, 19s.
Vol. XIII. (13 parts, each containing Abstracts), bound cloth (without Abstracts), 47s.
Vols. XIV. & XV. (12 parts, each containing Abstracts), bound cloth (without Abstracts), 23s.
Vols. XVI. & XIX. (8 parts each), bound cloth, 35s.
Vols. XVII., XVIII. & XXI. (7 parts each), bound cloth, 31s.
Vol. XX. & XXIV. (6 parts), bound cloth, 27s.

Most of the parts can be purchased separately, price 4s. by post 4s. 3d. Fellows can obtain the *Proceedings* (in parts) for their personal use at half the above prices.

ABSTRACTS OF PHYSICAL PAPERS
FROM FOREIGN SOURCES.

VOLS. I. (1895), II. (1896), and III. (1897), 15s. each; to Fellows, 7s. 6d. each.

Strong cloth cases for binding the "Proceedings," price 1s. 6d. each, post free.

BLAKESLEY, T. H. A Table of Hyperbolic Sines and Cosines. *Price 1s. 6d.; to Fellows, 9d.*

LEHFELDT, R. A. A List of Chief Memoirs on the Physics of Matter. *Price 2s.; to Fellows, 1s.*

Applications for the above Publications should be sent direct to
"THE ELECTRICIAN" PRINTING & PUBLISHING CO., LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET, LONDON.

CONTENTS.

PAGE

XVIII. The Laws of Variation of Resistance with Voltage at a Rectifying Contact of Two Solid Conductors, with Applications to the Electric Wave Detector. By D. Owen, B.A., B.Sc., Birkbeck College, London..... 173

XIX. The Latent Heat of Fusion of a Metal and the Quantum-Theory. By H. Stanley Allen, M.A., D.Sc., University of London, King's College..... 204

XX. Lenses for Light Distribution. By T. Smith, B.A. (From the National Physical Laboratory). 211

XXI. The Choice of Glass for Cemented Objectives. By T. Smith, B.A. (From the National Physical Laboratory). 220

XXII. The Correction of Chromatic Aberrations when the External Media are Dispersive. By T. Smith, B.A. (From the National Physical Laboratory)..... 235

XXIII. Note on the Use of the Auto-Collimating Telescope in the Measurement of Angles. By J. Guild, A.R.C.S., D.I.C., F.R.A.S. (From the National Physical Laboratory). 242

XXIV. The Viscosity of Colloidal Solutions. By Emil Hatschek..... 250

XXV. A Recalculation of Some Work on Diffusion by Dr. A. Griffiths and Others. By A. Griffiths, D.Sc..... 255